ALGEBRA AND FUNCTIONS GRADE BY GRADE

Kindergarten

At the kindergarten level, the Grade Level Expectations (GLEs) specify one stem that addresses patterns.

Patterns

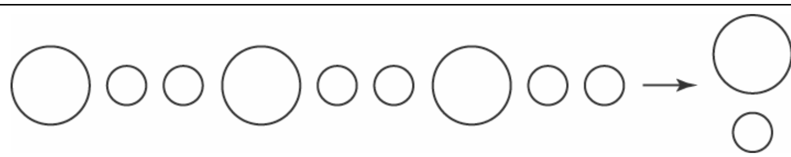
In kindergarten, students' work with repetitions, relations, and functions is focused on patterns. Students in kindergarten could be asked to identify and extend simple patterns. For example, they could be asked to identify what is repeating in a pattern consisting of a sequence of shapes, sounds, movements, colors, or letters. Once students have identified the repeating pattern they can be asked to extend the pattern to the next one, two, or three terms.

There are many different and correct ways of extending a given repeating pattern. When students present a pattern that is different from that expected by the teacher, the teacher should give students an opportunity to justify their response. This attempts to ensure that perfectly reasonable student-generated patterns are recognized.

In working with patterns, students in kindergarten should be asked to translate simple repeating patterns such as ABABAB across formats. For example, an ABABAB can be represented as *stomp clap stomp clap stomp clap.*

It is commonly assumed that at least three iterations of a pattern must be presented in order to ensure mathematical coherence. This is not strictly correct, but three or more iterations are useful in that it allows the students to figure out a pattern's repeating element more easily than do fewer than three iterations.

The following task is an example of a kindergarten level pattern activity. Each pattern keeps going like this: Color the shape that comes next.

a	
b.	

Grade 1

At the grade 1 level, the GLEs specify two stems that address (1) patterns and (2) equations.

Patterns

The first stem relates to patterns. Students in grade 1 could be asked to identify and extend simple patterns. For example, they could be asked to identify what is repeating in a pattern consisting of a sequence of numbers or shapes. They could also be asked to find a missing term in a pattern. For example, students might be asked to find the missing term in a sequence such as this one:

2, 4, 6, ___, 10

Once students have identified the repeating pattern they can be asked to extend the pattern to the next one, two, or three elements.

There are many different and correct ways of extending a given repeating pattern and unless students are asked to extend a pattern in a *particular way* extensions should not be automatically marked incorrect. When students present a pattern that is different from that expected by the teacher, the teacher should give students an opportunity to justify their response. This attempts to ensure that perfectly reasonable student-generated patterns are recognized.

In working with patterns, students in grade 1 should be asked to translate simple repeating patterns such as ABABAB across formats. For example, an ABABAB can be represented as *stomp clap stomp clap stomp clap.*

It is commonly assumed that at least three iterations of a pattern must be presented in order to ensure mathematical coherence. This is not strictly correct, but three or more iterations are useful in that it allows the students to figure out a pattern's repeating element more easily than do fewer than three iterations.

The following task is an example of a grade 1 activity:

Study each repeating pattern. Then color the shape that comes next.

a	
b.	

Equations

The second stem relates to equations. In grade 1, students should be asked to work with *open number sentences.*

A number sentence is *open* if it contains a placeholder such as a blank box, $\square$, or a letter to represent an unknown number. Therefore, an open number sentence is simply a friendlier name for an equation that contains an unknown.

Here is an example of an open number sentence that a first grade student could learn to solve:

What number goes in the box, $\square$, to make this number sentence true

$$4+5=\square+3$$

Work with open number sentences is designed to help students develop a robust understanding of the meaning of the equal sign.

In Grade 1, students could be given an open number sentence and asked, *What number must go in the box,* $\square$*, to make the number sentence true*? Or, asked to make up their own open number sentences.

In Grade 1 also, students could be given number sentences of the form:

15 + 16 = 16 +15.

These number sentences are not open because they do not contain an unknown and are numerical equations that are either true or false. Young children could be asked to say if a given number sentence is true or false, they could be asked to create their own true or false number sentences, and could be asked to determine if their peer-generated number sentences are either true or false.

When working with number sentences it is important to ask young children to read aloud their number sentence. In the video clips that accompany *Thinking Mathematically*, the researchers can be heard asking young children to read number sentences aloud before going to work on them. For example, one of the video clips shows Kevin, a kindergarten student, being asked to read the open number sentence, $4+5=\square+3$ aloud, and Kevin reads:

"4 plus 5 is the same as Hmm plus 3"

In response to Kevin's reading, the researcher says:

"Could you tell us what to put in the box, or in the Hmm, to make this number sentence true"

Furthermore, when working with open number sentences it is important that the unknown appears in different positions in the number sentence.

For example, students should experience a range of open number sentences such as:

$$6+7=5+\square$$

$$6+\square=7+3$$

$$4+5=\square+3$$

$$\square+5=4+3$$

Students in grade 1 will need to be accustomed to working with open number sentences that involve either addition or subtraction but not both.

Grade 2

At the grade 2 level, the GLEs specify two stems that address (1) patterns and (2) equations.

Patterns

The first stem relates to patterns. Students in grade 2 could be asked to identify and extend simple patterns. For example, they could be asked to identify what is repeating in a pattern consisting of a sequence of numbers or shapes. They could also be asked to find a missing term in a pattern. For example, students might be asked to find the missing term in a sequence such as this one:

4, 7,10, ___, 16

Once students have identified the repeating pattern they can be asked to extend the pattern to the next one, two, or three terms.

There are many different and correct ways of extending a given repeating pattern and unless students are asked to extend a pattern in a *particular way* extensions should not be automatically marked incorrect. When students present a pattern that is different from that expected by the teacher, the teacher should give students an opportunity to justify their response. This attempts to ensure that perfectly reasonable student-generated patterns are recognized.

The following task is an example of a pattern activity appropriate for grade 2 students:

An Add 28 Pattern

Complete the table below, and then use it to find the values of a for which $a + 29 > 32$.

a	1	2	3	4	5	6	7	8
$a + 29$								

Equations

The second stem relates to equations. In grade 2, students should be asked to work with *open number sentences* such as the following:

$$8 + 5 = \square + 4$$

A number sentence is *open* if it contains a placeholder such as a blank box, $\square$, or a letter to represent an unknown number. Therefore, an open number sentence is simply a friendlier name for an equation that contains an unknown.

Work with open number sentences is designed to help students develop a robust understanding of the meaning of the equal sign.

In grade 2, students could be given an open number sentence and asked, *What number must go in the box,* $\square$*, to make the number sentence true*? Or, asked to make up their own open number sentences.

In grade 2, students could also be given number sentences of the form:

$$19 + 17 = 19 + 17$$

These number sentences are not open because they do not contain an unknown and are numerical equations that are either true or false. Young children could be asked to say if a given number sentence is true or false, they could be asked to create their own true or false number sentences, and could be asked to determine if their peer-generated number sentences are either true or false.

Ask students to read their number sentence out loud. In the *Thinking Mathematically* videos the researchers can be heard asking young children to read number sentences aloud before going to work on them. For example, one of the video clips shows Kevin, a kindergarten student, being asked to read the open number sentence, $4+5=\square+3$ aloud, and Kevin reads:

"4 plus 5 is the same as Hmm plus 3"

In response to Kevin's reading, the researcher says:

"Could you tell us what to put in the box, or in the Hmm, to make this number sentence true"

The unknown appears in different positions in the number sentence. For example, students should experience a range of open number sentences such as:

$$8+5=\square+4$$

$$8+\square=9+4$$

$$8+5=\square+4$$

$$\square+5=9+4$$

Second grade students will need to be accustomed to working with open number sentences that involve either addition or subtraction but not both.

The following task is an example a numerical equations activity appropriate for grade 2 students.

Is It Magic?

In a *magic square* each row, column, and diagonal add up to the same number. Check whether these are magic squares.

9	8	13
14	10	6
7	12	11

40	5	30
15	25	35
20	45	10

11	12	15
16	12	8
9	14	13

Grade 3

At the grade 3 level, the GLEs specify two stems that address (1) patterns and (2) equations.

Patterns

The first stem relates to patterns. Students in grade 3 could be asked to identify and extend simple patterns. For example, they could be asked to identify what is repeating in a pattern consisting of a sequence of numbers or shapes or asked to find a missing term.

Once students have identified the repeating pattern they can be asked to extend the pattern to the next one, two, or three terms.

There are many different and correct ways of extending a given repeating pattern and unless students are asked to extend a pattern in a *particular way* extensions should not be automatically marked incorrect. When students present a pattern that is different from that expected by the teacher, the teacher should give students an opportunity to justify their response. This attempts to ensure that perfectly reasonable student-generated patterns are recognized.

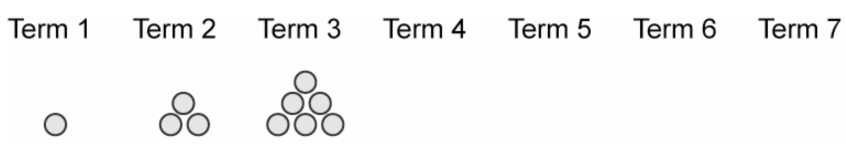
The following task is an example of a grade 3 patterns task:

Circle Patterns

The number of circles keeps growing as shown here.

What might the next two diagrams look like?

Complete the table to extend the pattern below.



Term number	1	2	3	4	5	6	7
Number of circles	1	3	6				

Equations

The second stem relates to equations. In grade 3, students should be asked to work with *open number sentences* such as the following:

$$8+9=\square+7$$

A number sentence is *open* if it contains a placeholder such as a blank box, $\square$, or a letter to represent an unknown number. Therefore, an open number sentence is simply a friendlier name for an equation that contains an unknown.

Work with open number sentences is designed to help students develop a robust understanding of the meaning of the equal sign.

In third grade students should be given an open number sentence and asked, *What number must go in the box,* $\square$*, to make the number sentence true*? Or, asked to make up their own open number sentences.

In third grade students could also be given number sentences of the form 15 + 16 = 16 +15. These number sentences are not open because they do not contain an unknown and are numerical equations that are either true or false. Young children could be asked to say if a given number sentence is true or false, they could be asked to create their own true or false number sentences, and could be asked to determine if their peer-generated number sentences are either true or false.

When working with number sentences it is important to ask young children to read aloud their number sentence. In the video clips that accompany *Thinking Mathematically*, the researchers can be heard asking young children to read number sentences aloud before going to work on them. For example, one of the video clips shows Kevin, a kindergarten student, being asked to read the open number sentence, $4+5=\square+3$ aloud, and Kevin reads:

"4 plus 5 is the same as Hmm plus 3"

In response to Kevin's reading, the researcher says:

"Could you tell us what to put in the box, or in the Hmm, to make this number sentence true"

Furthermore, when working with open number sentences it is important that the unknown appears in different positions in the number sentence. For example, students should experience a range of open number sentences such as:

$$8+9=7+\square$$

$$8+\square=7+10$$

$$8+9=\square+10$$

$$\square+9=7+10$$

Grade 3 students will need to be accustomed to working with open number sentences that involve either addition or subtraction but not both. However, throughout grade 3, students will need to be accustomed to working with open number sentences that involve just one of the operations of addition, subtraction, or multiplication. It is important to note that in grade 3, the nature of the work with equations does not change substantially from that seen in kindergarten through grade 2. The operations involved can be more complex in the later grades, but the importance of equations in K-3 is to provide the opportunity for students to learn what is meant by the equal sign and to understand properties such as the commutative property of addition.

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Grade 4

At the grade 4 level, the GLEs specify four stems that address (1) sequences, (2) functions and relations, (3) expressions, and (4) equations.

Sequences

The first stem relates to sequences. Students in grade 4 could be asked to identify, extend, and write the general rule for sequences presented in models, tables or situations. It is important to realize that when the rule for determining the next term is not made explicit, many different answers to next term problems may be reasonable and correct. Thus, when a student response does not match that of the teacher, student reasoning must be probed in order to judge the appropriateness of the student's response.

Functions and Relations

The second stem relates to relations and functions. Particularly foundational to the study of algebra and functions is the study of *co-variation,* or the study of how one quantity varies in relation to another quantity. For instance, in the following example you can investigate the relationship between the height of a stack of tokens and the number of tokens in the stack. Looking at how the height changes in relation to the number of tokens added or taken away from the stack means that you are investigating the co-variation of the height of a stack of tokens and the number of tokens in the stack:

Suppose each token is 2 mm thick. Suppose also that the number of tokens in a stack is represented by the letter n. Then the height, H, of a stack of tokens is given by:

$$H = 2n$$

In grade 4, students begin their study of functions and relations by exploring how two different quantities vary in relation to each other. When introducing the concept of co-variation to young children it is important that they can "see" in their mind's eye how one quantity actually changes in relation to another.

During grade 4 the focus needs to be on varying quantities that conform to the form $y = kx$ because the GLE M(F&A)-4-2 requires that a student:

> Demonstrates conceptual understanding of linear relationships ($y = kx$) as a constant rate of change by identifying, describing, or comparing situations that represent constant rates of change.

Given a stack of tokens, coins or other common object that stack, the idea of constant rate of change can be made accessible to young children. For example, given an actual stack of tokens such as one of those pictured above, students can see that every time one token is added to the stack the height changes by a constant amount. This constant amount gives the rate of change. In the context of stacking identical tokens it makes sense that the rate of change is “constant”.

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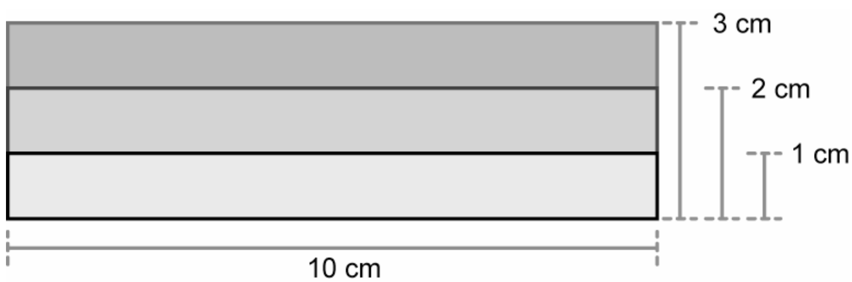
During grade 4, it is important to introduce a number of contexts that illustrate a constant rate of change and ask students to use models and tables and to explain the relationships between the two quantities in words and in symbols. For example, this series of growing triangles represents another context that will be accessible to most fourth grade students:

Students in grade 4 can be asked to explain how the triangle pattern is growing. They can also be asked if the triangles are growing by the same amount from one pattern to the next.

Another context that is appropriate for grade 4 is that of rectangles with a fixed length:



This diagram shows a series of rectangles and each rectangle has a fixed length of 10 centimeters. The heights of the rectangles vary. For example, the heights of the rectangles shown in the diagram are 1, 2, and 3 centimeters, respectively. It is possible to use the measurements given in the diagram to find the area of each rectangle.

These could be represented in a table as follows:

Height	1	2	3	4	5	6
Area	10	20	30	40	50	60
Change as height increases by 1	10		10		10	

Thus this activity allows students to explore how the area of a rectangle with a fixed length changes as its height changes. From the table you can see that the rate of change is constant.

Contexts such as the token, the growing triangles, or the rectangles with a fixed length are all sufficiently concrete for young students to be able to “see” and count how one quantity is changing in relation to another quantity. Further, when students study functions using the context of the rectangles with fixed length, they are also studying area—two-dimensional measurement. Thus students are being given the opportunity to consolidate and reinforce area as they undertake a study of co-variation and constant rate of change:

Expressions

The third stem relates to expressions. Grade 4 students are expected to use letters or symbols (i.e., x and y or □ and ▯) to represent unknown quantities and to write linear expressions that involve any *one* of the four operations. Students are also expected to find the value of a simple linear expression by evaluating it for given values of the unknown. When students are assigned the task of evaluating simple linear expressions it is important that the computational load of the assignment is restricted to just one operation and positive whole numbers.

In grade 4, students can be expected to simplify numerical expressions involving parentheses. For example, students could be asked to evaluate the following:

$$3 + 5 \bullet (3 + 4)$$

Students’ work with expressions can be integrated with their work with functions and equations. For example, in the stack of tokens task, students found that the height of a stack of n tokens was given by the formula:

$$H = 2n$$

Students can be asked to find the height of a stack of 24 tokens.

To respond to this prompt, students need to substitute 24 for n in the formula,

$H = 2n$, *giving:*

$$H = 2 \bullet 24$$

Thus this prompt requires that they *evaluate* an expression for a given value of a variable.

Equations

The fourth stem relates to equations. During grade 4, students work with the linear equations specified in GLE *M*(*F*&*A*)-4-4.

> …and by solving one-step linear equations of the form $ax = c$ and $x + b = c$, where a, b, and c are whole numbers and $a \neq 0$.

Throughout grade 4, the nature of the work with equations becomes increasingly more complex. It is during this grade that students begin to use letters or other symbols to represent unknown numbers in equations.

Students work with open number sentences that involve any one of the four operations of addition, subtraction, multiplication, or division.

For example, students need to be able to solve equations such as:

$4x = 16$	the unknown is multiplied by a positive whole number
$x – 3 = 10$	subtraction of positive whole numbers
$10 – x = 10$	subtraction of zero
$x + 4 = 27$	addition of whole numbers
$10 + 0 = 10$	addition of zero (zero property)

However, students in grade 4 would not be expected to solve equations that include negative numbers such as: $10 – x = -15$

Students' work with equations can be integrated with their work with functions. For example, in the stack of tokens task students found that the height of a stack of n tokens was given by the formula:

$$H = 2n$$

Students can be asked to find out how many tokens there are in a stack of tokens that is 50 mm high.

To respond to this prompt students need to substitute 50 for H in the formula, $H = 2n$ giving

$$50 = 2n$$

Thus, this prompt requires that they set up and solve a grade level appropriate equation in one unknown.

Grade 5

At the grade 5 level, the GLEs specify four stems that address (1) sequences, (2) functions and relations, (3) expressions, and (4) equations.

Sequences

The first stem relates to sequences. Students in grade 5 should be asked to identify, extend, and write the general rule for linear sequences presented in models, tables or situations. It is important to realize that when the rule for determining the next term is not made explicit, many different answers to next term problems may be reasonable and correct. For example suppose students were given this sequence before being asked to write the next two terms.

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5, 10, 15, 20

It seems obvious that the next terms are 25 and 30.

However, the next terms could be 15, 10, or 5 because the rule for determining the sequence was not made explicit. Thus if students were to give an answer other than 25 and 30, the teacher should explore students' reasoning to investigate their thinking.

Functions and Relations

The second stem relates to relations or functions. One of the big mathematical ideas related to constant rate of change and $y = kx$, is per unit quantity. Indeed, *per unit quantity* is one of the big mathematical ideas associated with the study of two quantities that vary in relation to each other. Per unit quantity is also known as the constant of proportionality in proportional relationships or the k in $y = kx$.

It is an extremely useful measure that shows up in many different forms in our lives and in mathematics.

Real world Applications of Per Unit Quantity

a. Population density: Suppose you were to calculate a good estimate for the number of square meters of land in the United States and then calculate a good estimate for the number of people in the United States. You would get the per unit quantity, *number of people per square meter*, if you were to divide the number of square meters by the number of people:

$$\frac{\text{number of square meters in U.S.}}{\text{number of people in U.S.}} = \text{number of people per square meter}$$

b. Unit price: If you were to buy a 5 pound watermelon for $3.50, you could calculate the *per unit price* by dividing the total price by the number of pounds.

   A per unit quantity is a type of rate that allows you to investigate *rate of change*. When the relationship between two varying quantities can be characterized by a per unit quantity, then the rate of change of one quantity in relation to the other will be constant.

c. An example of how per unit quantity may be used to compare complex phenomenon such as agricultural yield:

Suppose that Farm *A* uses 800 square meters to grow strawberries and produces 3600 kilograms of strawberries.

Suppose that Farm *B* uses 1000 square meters to grow strawberries and produces 4800 kilograms of strawberries per harvest.

Clearly Farm *B* produces more strawberries than Farm *A*, but Farm *B* also uses more land for growing strawberries than Farm *A*. In order to compare the yield of these two farms you could calculate the amount of strawberries that is produced per square meter in each of the farms. Or, you could compare the yield of the farms by computing the amount of land required per kilogram of strawberries in each of the farms.

For example:

Kilograms per square meter		Square meter per kilogram
Farm A $\frac{800}{3600} = 0.22$ kg		Farm A $\frac{3600}{800} = 4.5$ m^2
Farm B $\frac{1000}{4800} = 0.21$ kg		Farm B $\frac{4800}{1000} = 4.8$ m^2

From these calculations you can see that Farm *A* produces more strawberries per square meter than Farm *B*. Also you can see that to produce a kilogram of strawberries Farm *A* has to set aside a smaller amount of land. Thus you can see that Farm *A* produces a better yield than Farm *B*.

d. The concept of speed:. Speed tells the distance covered per unit of time or tells the time taken to cover a particular unit of distance. For example, we talk about the speed of a train in terms of miles per hour or kilometers per hour. Here is an example:

The Acela Express train travels 205 miles in 3 hours. The speed of the Acela can be found using this formula:

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

In this formula, distance and time can be measured in any appropriate unit and the resulting per unit quantity, called “speed,” will be specified in terms of these units. For example the speed of a train is usually given in miles per hour while the speed of a tortoise is usually given in inches per minute.

In grade 5, students must learn to use letters (such as *x* and *y*) to represent varying quantities. There are many real and worthwhile contexts that are accessible to students in Grade 5 that can be explored in order to provide students with the opportunities to learn how to represent the relationship between two quantities using letters. For example, given the formula:

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

Students will be expected to express it using letters to represent the quantities as follows:

$$s = \frac{d}{t}$$

Expressions

The third stem relates to evaluating expressions created by letters rather than expressions created by blank boxes or other symbols. The expressions that students are expected to evaluate can include any two of the four operations.

Equations

The fourth stem relates to linear equations. In grade 5, students are expected to solve certain types of linear equations. For example, students need to be able to solve equations such as:

$7x = 30$	One expression is a product of an unknown and a positive whole number. The other expression is another positive whole number.
$x - 7 = 35$ or $x + 7 = 35$	A positive whole number is subtracted or added to an unknown with coefficient 1.
$\frac{x}{3} = 21$	One of the expressions is an unknown with a unit fraction as its coefficient. The other expression is a positive whole number.

In addition to solving equations, the GLEs indicate that students can be asked to identify the solution to a more complex linear equation from among three or four given solutions. In this context, a more complex linear equation is one that involves more than one of the four operations of addition, subtraction, multiplication, and division. For example, students could be asked to say which of the numbers 2, 3, 4, or 5 is a solution to the equation $2x + 3 = 11$.

At the grade 6 level, the GLEs specify four stems that address (1) sequences, (2) functions and relations, (3) expressions, and (4) equations.

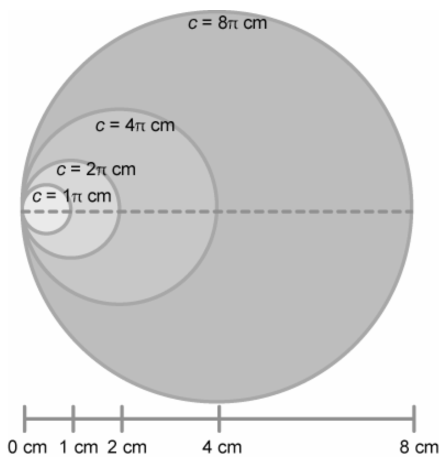
Sequences

The first stem relates to sequences. Students in grade 6 should be asked to identify, extend, and write a general rule for a linear sequence presented in models, tables, graphs, or situations. It is important to realize that when the rule for determining the next term is not made explicit, many different answers to next term problems may be reasonable and correct. In these cases, student thinking should be explored.

Functions and Relations

The second stem relates to relations and functions. Grade 6 is the time to introduce most students to the special type of co-variation that is called *direct variation,* a type of co-variation that characterizes *Proportional Relationships*.

If one varying quantity y is a constant multiple of another varying quantity x, then quantity y is *directly proportional* to the quantity x. For example, the relationship between the circumference and diameter of a circle is a *directly proportional* relationship.



In this diagram there are four circles with diameters of 1, 2, 4, and 8 centimeters respectively. The circumference of a circle c and its diameter d are related by the formula:

$$c = \pi d$$

You can see that in this formula, $c = \pi d$, the circumference c is a constant multiple of the diameter *d.* That is, the circumference of any circle is π times the diameter.

Another way to think about this is to find the *ratio* of the circumference to the diameter of the circles shown above. These are shown in this table, below:

Diameter	1	2	4	8
Circumference	$1 \bullet \pi$	$2 \bullet \pi$	$4 \bullet \pi$	$8 \bullet \pi$
Ratio $\frac{c}{d}$	$\frac{\pi}{1} = \pi$	$\frac{2\pi}{2} = \pi$	$\frac{4\pi}{4} = \pi$	$\frac{8\pi}{8} = \pi$

You can see that the ratio of the circumference to the diameter is π. This is true for any circle. So, you can say that the circumference is proportional to the diameter.

The constant π that shows up here in division as the constant ratio, and in multiplication as the constant multiple, is called the *constant of proportionality*.

In any relationship where x is proportional to y, if the quantity x is doubled, tripled, or quadrupled etc., then the quantity y is doubled, tripled, or quadrupled etc., respectively.

To better understand this, consider the circle diagram or the table above. You can see that when the diameter is doubled the circumference is doubled, as well. That is, when $d = 1$, $c = \pi$, but when $d = 2$, $c = 2\pi$. Similarly if you quadruple the diameter you automatically quadruple the circumference. From the diagram and the table above you can see that when $d = 2$, $c = 2\pi$, but when $d = 8$, $c = 8\pi$.

Here is a summary of the main ideas involved in proportional relationships:

- If two quantities x and y vary in such a way that y *increases* by a factor of 2, or 3 etc., as x *increases* by a factor of 2, or 3 etc., then you can say that y is proportional to x.
- If two quantities x and y vary in such a way that y *decreases* by a factor of 2, or 3 etc., as x *decreases* by a factor of 2, or 3 etc., then you can say that y is proportional to x.
- If two quantities x and y vary in such a way that there is a constant ratio between corresponding values of x and y, then you can say that y is proportional to x.
- If two quantities x and y vary in such a way that one is a constant multiple of the other, then you can say that y is proportional to x.

Both the constant ratio and the constant multiple associated with proportional relationships can be used to develop students' understanding that every proportional relationship can be expressed in the form:

$$y = kx.$$

Or, that every proportional relationship can be expressed as:

$$y = \text{Fixed number} \bullet x$$

When students are asked to write an equation to express the relationship between two quantities *x* and *y*, students will learn that when *x* is proportional to y, the equation will take the form, y = fixed number • x, or more formally $y = kx$.

In $y = kx$, x and y represent the varying quantities and k represents a fixed number, or what is more formally called the constant of proportionality.

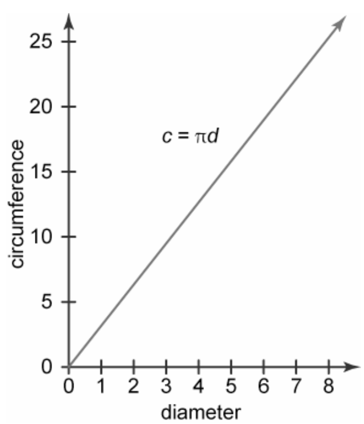
When students are asked to graph the relationship between two quantities, *x* and *y*, they will learn that when *x* is proportional to *y* the graph will take the form of a straight line that passes through the point $x = 0$ and $y = 0$.

For example, suppose that you represented the above table of values as a graph. You would find that the graph is a line which passes through (0,0).

It makes sense that the graph of the circumference and diameter is a line, because if you double the diameter you double the circumference. It also makes sense that the line will pass through the point (0,0) because if the diameter is zero then the circumference will be zero.

In the graph below, the circumference, c, is represented on the vertical axis and the diameter, *d*, is represented on the horizontal axis. You can see that the graph is a line that passes through (0.0).

Graph of Diameter against Circumference



The graph is a line.

The line passes through the point (0,0).

The slope of the line is π, a constant ratio or the constant of proportionality.

A line passing through the origin, (0,0) is a characteristic feature of the graph of any proportional relationship.

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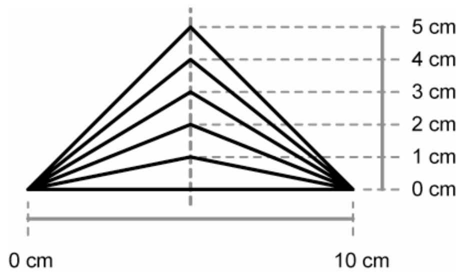
In the example below, students are given a set of triangles where the base length is fixed. So, although each triangle has the same base each has a different height. Given this context students are asked to determine if the height of a set of triangles with a fixed base is proportional to the area.

The two varying quantities:

the area of triangles with base length 10 centimeters,

the height of each triangle varies

For any triangle with a fixed base, is the height proportional to the area?



Expressions

The third stem relates to expressions. In grade 6, students are expected to work with expressions involving any of the four operations. Also students can be expected to evaluate more complex expressions including those involving more than one variable.

Equations

The fourth stem relates to equations. In grade 6, students are expected to solve linear equations of the form: $ax + b = 0$. Students are expected to solve linear equations, (note- such as should be pulled up to this line)Such as:

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- $7x + 3 = 30$ or $7x – 3 = 30$ the coefficient of the unknown is a *positive* whole number and all other constants involved are whole numbers.

In mathematics textbooks and in mathematics education literature, equations of the form, $ax + b = 0$, are sometimes expressed as $ax + b = c$ or even as $ax \pm b = c$. There is a historical reason for writing the general form of a linear equation, $ax + b = 0$ as $ax + b = c$. However, $ax + b = 0$ is considered the standard way of describing either $ax \pm b = c$ or $ax + b = c$, and is sufficient for our purposes.

Given a linear equation in two unknowns (i.e., $y = mx + b$), students are expected to be able to find the value of y for given values of x.

At the grade 7 level, the GLEs specify four stems that address (1) sequences, (2) functions and relations, (3) expressions, and (4) equations.

Sequences

The first stem relates to sequences. Students in grade 7 should be asked to identify, extend, and write a general rule for a linear or non-linear sequence presented in models, tables, graphs, or situations. It is important that the rule for determining the next term is made explicit when students are asked to extend and generalize a sequence. When it is not made explicit, many different answers to next term problems may be reasonable or correct. In these cases, student-thinking should be explored.

Functions and Relations

The second stem relates to relations and functions. In grade 7, students begin to link their work with proportional relationships to linear functions. A linear function can always be represented by the formula:

$$y = kx + b,$$

Here, x and y represent variables and b and k represent parameters that remain fixed for a particular linear function.

When the linear function also represents a proportional relationship, the formula becomes:

$$y = kx$$

This is because in a proportional relationship b must be equal to zero.

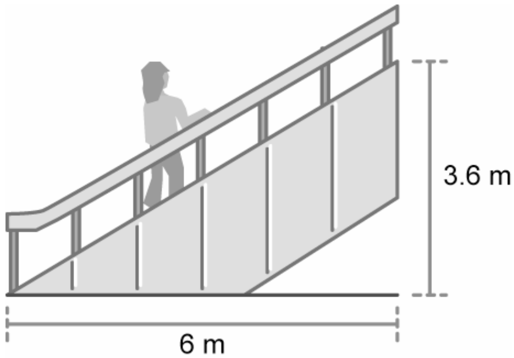
In a proportional relationship the parameter k represents the constant of proportionality.

The graph of $y = kx$ is a line that passes through the point (0,0). The parameter k gives the slope of the graph. When $k > 0$ the line slopes upward, so as x increases, y increases. When $k < 0$ the line slopes downward, so as x decreases y decreases.

Many useful contexts can be used to investigate linear functions. Here is an example:

Escalator

The picture shows a woman going up an escalator.



If you let *x* represent the horizontal distance traveled in meters and *y* represent the vertical distance traveled in meters the relation between *x* and *y* can be investigated using the following table of values:

Horizontal distance	*x*	0	1	2	3	4	5	6
Vertical distance	*y*	0	0.6	1.2	1.8	2.4	3.0	3.6

Given this situation and these data, students can be asked to determine if a proportional relationship exists between the horizontal distance, *x*, and the vertical distance, *y.* Students might also be asked to express the vertical distance in terms of the horizontal distance and, thus, express *y* in terms of *x*

To decide whether or not the relationship between x and y is proportional students will have to decide if the vertical distance is a constant multiple of the horizontal distance. This involves determining whether the relationship can be expressed as follows:

$$y = \text{fixed number} \bullet x$$

You can see from the table that each value of *y* is 0.6 times the corresponding value of *x*. Thus, the relationship can be expressed as follows:

$$y = 0.6x$$

Therefore the relationship between the horizontal distance and the vertical distance is indeed a proportional one. In this situation 0.6 is the constant of proportionality.

Expressions

The third stem relates to expressions. In grade 7, students are expected to work with expressions involving any of the four operations and whole number exponents. Also students can be expected to evaluate more complex expressions including those involving more than one variable and whole number exponents.

Given the escalator problem, students could be asked to find the vertical distance traveled when the horizontal distance is 8 meters. Thus, students will have to evaluate the expression $0.6x$ when $x = 8$.

Equations

The fourth stem relates to equations. Students will need to be accustomed to solving linear equations of the following type:

> Equations of the form, $ax + b = cx + d$, with the following restrictions: (At Grade 7 the coefficients of x in all equations should be positive whole numbers. Constant terms such as b and d should be restricted to the set of integers. When the coefficient of x is a rational number, the unknown should appear on just one side of the equation. When the rational coefficient of x is expressed as a fraction, the fraction should be restricted to unit fractions.)

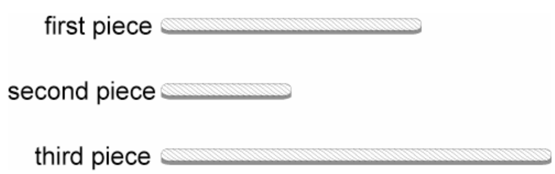
Given the escalator problem, students could be asked to find the horizontal distance traveled when the vertical distance traveled is 4 meters. Thus, students will have to set up and solve the following equation:

$$4 = 0.6x$$

In addition to solving equations, students will be expected to translate a word problem into an equation. The following task is an example of such a problem.

Cutting Rope

A 120-foot long rope is cut into 3 pieces. The first piece is twice as long as the second piece of rope. The third piece of rope is three times as long as the second piece of rope. What is the length of the longest piece of rope?



A model solution to the problem is given below:

Let the length of the second piece of rope be x.

Then the first piece of rope is of length $2x$.

The third piece of rope is of length $3x$.

You know that the sum of the three pieces of rope is 120 feet.

Thus, the *first piece* + *second piece* + *third piece* = 120.

It follows from this that $x + 2x + 3x = 120$. (a linear equation)

Solving this gives $6x = 120$.

Or $x = 20$.

The length of the longest piece of rope is 60 cm.

Grade 8

At the grade 8 level, the GLEs specify four stems that address (1) sequences, (2) functions and relations, (3) expressions, and (4) equations.

Sequences

The first stem relates to sequences. Students in grade 8 are expected to identify and extend to specific cases a variety of linear or non-linear sequences that are presented in models, tables, graphs, or situations. It is important to realize that when the rule for determining the next term is not made explicit, many different answers to next term problems may be reasonable or correct. When students produce a next term that differs from the teacher's expected answer, students reasoning should be probed so as to avoid inadvertently dismissing a perfectly reasonable response.

In grade 8, students should be given the opportunity to provide a formula to generalize a linear sequence that is presented in a model, a table, a graph, or in a situation. Students are also expected to generalize nonlinear sequences that are presented in models, tables, graphs, or situations, but students can do so using words rather than being required to provide a formula.

Functions and Relations

The second stem relates to relations and functions. By the end of grade 8, students need to be able to use linear relationships, $y = mx + b$ to solve problems. Students should be able to represent linear relationships in tables, graphs, formulas and words, and translate among these representations. Students should be able to find the slope and intercepts of the graph of a linear relationship and to identify the slope of the graph as the constant rate of change of the linear relationship. Given a problem situation, students should be able to represent it as a graph, a formula, or a table before finding the slope and y-intercept of the graph. Students will use these representations to interpret the original problem.

Students should be able to distinguish between linear and nonlinear relationships and hence between constant and varying rates of change. In addition, students should be able to describe how one quantity varies in relationship to another quantity.

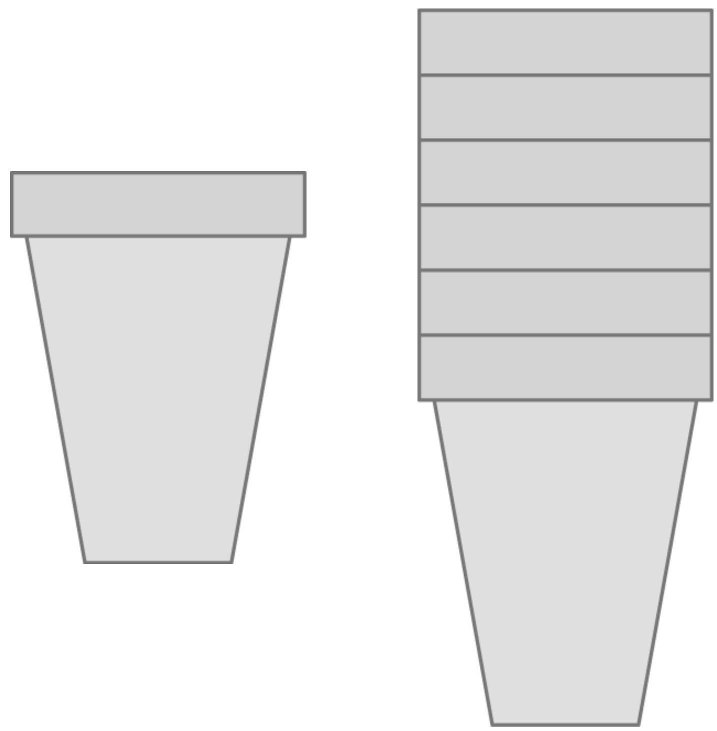
Grade 8, students are expected to re-express formulas such as $d = rt$ in terms of t, giving $\frac{d}{r} = t$.

The following task is an example of a grade 8 appropriate problem.

Paper Cups

The diagram below shows a drawing of one paper cup and six paper cups stacked together.

The cups are shown half size.



1 cup

6 stacked cups

Create a formula that gives the height of a stack of cups in terms of the number of cups in the stack. Define your variables and show *how* you created your formula.

Represent this situation in a graph.

Find the slope of the graph.

Interpret the slope of the graph in terms of the stack of paper cups.

Expressions

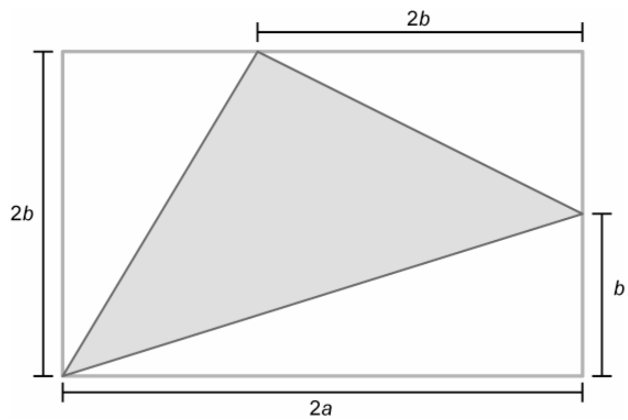
The third stem relates to expressions. In grade 8, students are expected to evaluate expressions involving one or more of the four operations, rational numbers, whole number exponents, or square roots. Students will also be expected to recognize or write a given expression in an equivalent form.

In the context of the stack of *Paper Cups* problem, shown above, students could be asked to find the height of a stack of 80 of the cups . In response to this students would need to take the expression that they created for the height of a stack of n cups and evaluate it for the value of $n = 80$.

The following task is an example of a problem that students could be asked to solve in Grade 8:

Shaded Triangle

The triangle is inside a rectangle. Create an expression for the area of the shaded triangle.



Equations

The fourth stem relates to equations. Students will need to be accustomed to solving linear equations in one unknown that are of the following form:

$$ax + b = cx + d, \text{ where } a, b, c, \text{ and } d \text{ are integers.}$$

In the context of the stack of *Paper Cups* problem, shown above, students could be asked to find the number of these paper cups that would fit in a container 1 meter high. In response to this, the student would need to use the formula that they created to express the height of a stack of cups in terms of the number of cups. They would then set up and solve a linear equation with this formula. For example, suppose that the formula for the height of this stack of cups in terms of the number of cups is as follows:

$$y = 1.5n + 14$$

Then, to find the number of cups that would fit in a container 1 meter high, students would have to set up and solve this equation:

$$100 = 1.5n + 14$$

In addition to solving linear equations in one unknown, students will be expected to solve problems in context that involve systems of linear equations in two unknowns.

Given a linear or nonlinear equation in two unknowns, *x* and *y,* students will be expected to find the value of *y* for given values of *x*. For example, students could be asked to find the value of y when x is 9, given the following non linear equation:

$$y = 7\sqrt{x} + 2x$$

Grade Span 9-10

At the 9-10 grade span, the GSEs specify four stems that address (1) sequences, (2) functions and relations, (3) expressions, and (4) equations.

Sequences

The first stem relates to sequences. Students in grades 9-10 are expected to be able to identify, extend, and generalize a variety of sequences of numbers. Students could be asked to determine a next term in a given sequence and give a reason for their choice. However, the rules for determining the next term should be accurately described because, without explicit rules, many different answers to next term problems may be reasonable and correct. It should be noted that sequences drawn from number and geometry generally have explicit rules, but patterns observed in collected data generally do not.

Functions and Relations

The second stem relates to functions and relations. In grades 9-10, students need to be given the opportunity to learn what a function is, and be able to distinguish between a function and a relation. Students also need to develop a thorough and sophisticated understanding of linear functions and apply their understanding to interpret, model, and solve problems. This entails understanding that the essential characteristic of linear functions is that they have a *constant rate of change*. Students must then also understand *why* the graph of a linear function is a straight line.

Students in grades 9-10 need to become familiar with a common algebraic representation of a linear function: $y = mx + b.$ In this formula the independent variable is $x,$ the dependent variable is y, the parameter m represents the rate of change or *slope* of the linear function, and the parameter b represents the value of the dependent variable y when the independent variable x equals 0. That is, when $x = 0$, $y = b$.

Students need to understand that in the graph of $y = mx + b$, the value of the parameter m is the slope of the line, and that this makes sense because the slope of the line is the ratio of the change in y to the change in x over any stretch of the line.

Students also need to understand that in the graph of $y = mx + b,$ the parameter b is the y-coordinate of the point where the line intersects the vertical axis, and that this makes sense because the line crosses the vertical axis when $x = 0$ such that $m \bullet 0 + b = b$. Thus, the line crosses the y-axis at the point $(0, b)$.

Students need to understand that linear functions can be represented, in some sense, with a table of values: The linear function $y = 4 - 2x$, for example, can be represented, in part, by the following table:

x	–2	–1	0	+1	+2	+3
Y	8	6	4	2	0	–2

However, to know that the table represents a *linear* function, or even a function, requires more information than just the values in the table. Other, non-linear, functions may also have the six values in this table.

Students need to be able to notice that the pairs in the table such as (1, 2) are coordinates of points that lie on the graph of $y = 4 - 2x$.

They also need to *see* the parameter *b* in this table by looking for the *y*-value corresponding to $x = 0$.

Students need to be able to find the rate of change from a table of values, (which is understood to represent some linear function), by calculating the ratio of the change in *y* and the change in *x*. Here is an example that is based on the two points (2, 0) and (1,2), that lie on the line $y = 4 - 2x$:

$$\frac{0-2}{2-1} = \frac{-2}{1} = -2$$

Here is another example that is based on the two points (–1, 6) and (0, 4), that lie on the line $y = 4 - 2x$:

$$\frac{4-6}{0-(-1)} = \frac{-2}{1} = -2$$

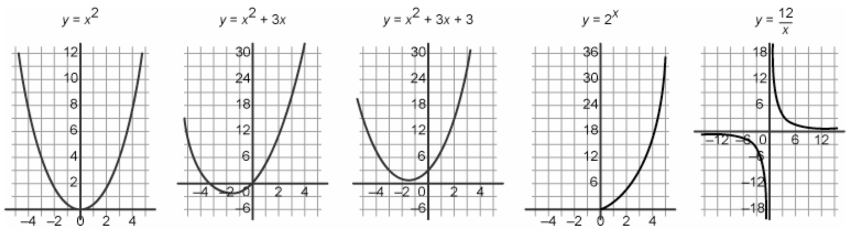
Students need to thoroughly understand that if given as few as two pairs of values that are known to lie on the graph of some linear function, they can then determine all other pairs of values for this linear function.

In grades 9-10 students need the opportunity to work with examples of non-linear functions as well as to understand that all non-linear functions have variable rates of change.

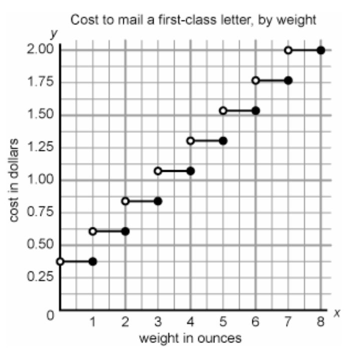
For example, students need to understand that each of the following equations is an example of a function, but none is an example of a linear function:

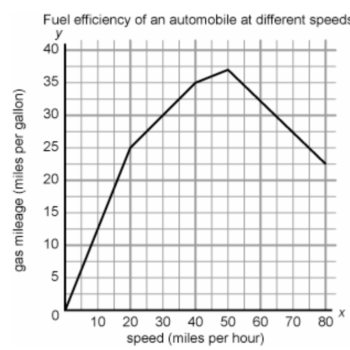
$y = x^2$ $\quad y = x^2 + 3x$ $\quad y = x^2 + 3x + 3$ $\quad y = 2^x$ $\quad y = \frac{12}{x}$

In each of these functions the rate of change is not constant, as seen from looking at the graph of each function. None of the graphs is that of a straight line.

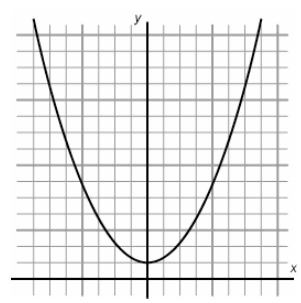


Students must also recognize certain discontinuous functions as non-linear, for example, the step function and the piecewise linear function shown below. Note that the step function and the piecewise linear function do have constant rates of change at various *intervals*, but that any change in y is not a constant multiple of the corresponding change in x.





Specifically, students need to be familiar with quadratic functions and their graphs. Students need to know that the general form of a quadratic function is given by the formula $ax^2 + bx + c$ and that the graph of a quadratic function is a parabola.


A parabola

Expressions

The third stem relates to expressions. In grades 9-10, students need to be given the opportunity to develop a thorough and sophisticated understanding of linear expressions. This entails understanding that a linear function of the form $y = f(x)$ can be evaluated at a specific value of x, the independent variable, to yield a corresponding value of y, the dependent variable.

Students need to be able to manipulate algebraic expressions so as to express them in equivalent forms. They need to be able to evaluate polynomial or rational expressions, or expressions involving integer exponents, square roots, or absolute values.

Students need to be able to solve problems involving algebraic expressions and translate problem situations into algebraic expressions.

Equations and Inequalities

The fourth stem relates to equations and inequalities. In grades 9-10, students need to be given the opportunity to develop a thorough and sophisticated understanding of linear equations in one variable. This entails knowing that given a linear function of the form $y = f(x)$, a linear equation in x can be determined by specifying a value of y, the dependent variable. The general from of a linear equation is $0 = f(x)$.

Students need to be able explain that the solution to the linear equation $f(x) = g(x)$ is the x–value of the intersection of the graphs of $f(x)$ and $g(x)$. Also, students should be able to explain that the solution to the linear equation $h(x) = 0$ is the x–value of the intersection of the graphs of $h(x)$ and the x–axis.

Students need to know that when the expression specifying the right side of a linear equation is equivalent to the expression on its left side the equation is called a linear identity.

Students must be familiar with systems of linear equations. They must know what is meant by a system of linear equations in two variables. They also need to understand why the coordinates of the point of intersection of the lines represented by the equations are the solutions to the system. In addition, students need to be able to solve simultaneous linear equations in two variables using any algebraic method.

Students must also be familiar with linear inequalities. Students need to know what is meant by a linear inequality in one or two variables. They need to understand that when both sides of an inequality are multiplied or divided by a negative number the inequality is reversed, and that all other basic operations, when applied to both sides, preserve the inequality. Students need to understand why the graph of a linear inequality in two variables is a half plane and be able to solve linear inequalities in one variable.

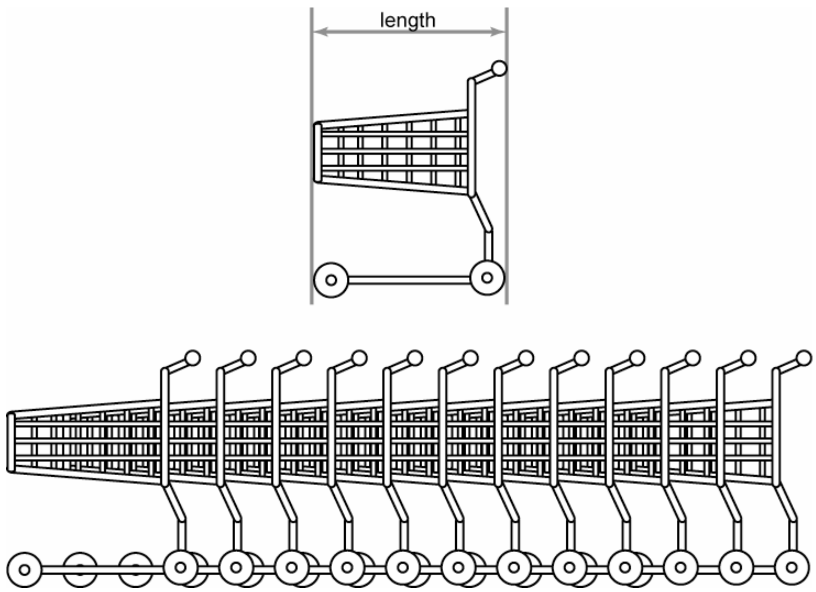
Students need to be able to solve contextualized problems by translating the context into a linear equation in one variable, a linear inequality in one variable, or into a system of linear equations in two variables and then solve the problem algebraically or graphically.

The following *Balanced Assessment* task is an example of a problem appropriate for students in grades 9-10.

Supermarket Carts

The diagram below shows a drawing of a single shopping cart.
It also shows a drawing of 12 shopping carts that have been "nested" together.

The drawings are accurately scaled to $\frac{1}{24}^{\text{th}}$ *real size*.



1. Express the length of a “nested” row of shopping carts in terms of the number of carts
2. What is the length of a “nested” row of 40 carts?
3. How many carts will fit in a space 12 meters long?

The following task is another appropriate problem:

Stopping Distances

Jamal had to learn the stopping distances of a car at various speeds for his written driving test. His driving manual contains the following table of values:

Speed, *v* (miles per hour)	0	30	60
Stopping Distance, *d* (feet)	0	60	240

Graph the data in the table:

Write an equation for Stopping Distance in terms of Speed.

If Jamal is driving at 80 miles per hour, what is his stopping distance?

Grade Span 11 –12

At the 11-12 grade span, the GSEs specify four stems that address (1) sequences, (2) functions and relations, (3) expressions, and (4) equations.

Sequences

The first stem relates to sequences. In grades 11-12, students need to be given the opportunity to develop a thorough and sophisticated understanding of arithmetic and geometric sequences. Specifically, students will be expected to identify arithmetic and geometric sequences and find the nth term before using this generalization to find a specific term.

Functions and Relations

The second stem relates to functions and relations. In grades 11-12, students need to be given the opportunity to develop a thorough and sophisticated understanding of functions through representing and analyzing them. Students will need the opportunity to be able to recognize properties of functions and the characteristic properties of families of functions. In grades 11-12, students will need to apply their knowledge of functions to interpret and understand situations, design mathematical models, and solve problems in mathematics as well as in natural and social sciences.

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Students need the opportunity to analyze the essential characteristics of polynomial, rational, and exponential functions. This work should include a study of domain, range, intercepts, increasing and decreasing intervals, and rates of change.

Students need the opportunity to represents functions numerically, algebraically, graphically, and verbally. They also need to identify the properties of a function from these representations and transfer information from one representation to another.

Students need the opportunity to transform the graphs of polynomial, rational, and exponential functions. This study should include vertical and horizontal shifts, stretches and compressions, and reflections across vertical and horizontal axes.

The following problem is an example of a functions task for grades 11-12:

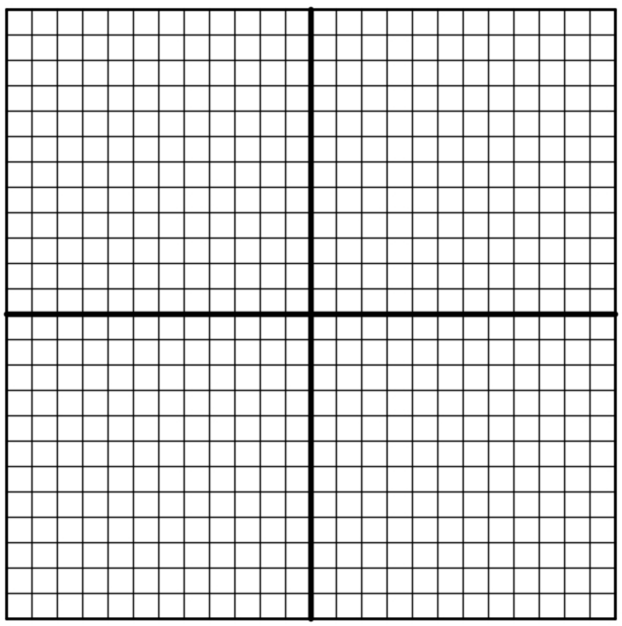
Analyzing Functions

Consider the function $f(x) = 5^x$

(a) State the domain and range of the function

(b) Write an equation for $g(x) = f(-(x-2))$.

(c) Graph $f(x)$ and $g(x)$ on the axes below. Describe how one could obtain a graph of $g(x)$ from the graph of $f(x)$.



Expressions

The third stem relates to expressions. In grades 11-12, students need to be given the opportunity to develop a thorough and sophisticated understanding of polynomial, rational, and exponential expressions. This entails understanding that a function in *x* can be evaluated at specific values of *x* in order to yield a corresponding value of *y*.

Students need the opportunity to become adept at polynomial arithmetic by adding, subtracting, multiplying and dividing polynomial and rational expressions. In addition, students should practice simplifying complex fractions and expressions including radicals.

Students need to be able to factor quadratic expressions and higher degree polynomial expressions. They also need to be able to know and represent geometrically the following identities:

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$
$$(a + b)(a - b) = a^2 - b^2$$

Fluency with these identities enables students to handle the levels of algebra that underpins advanced mathematics.

Students need to understand the properties of logarithmic expressions (e.g. $\log_a b^n = n\log_a b$, $a^{\log_a b} = b$) and to be able to convert between logarithmic and exponential forms of an expression.

For example, students could be given the following task:

> Rewrite the expression $2\log_3 x + \log_3 y - \log_3 2$ as a single log expression.

Equations and Inequalities

The fourth stem relates to equations and inequalities. In grades 11-12, students need to be given the opportunity to develop a thorough and sophisticated understanding of quadratic equations, quadratic inequalities, and systems of equations and inequalities in two variables.

In this grade span, students need to be able to solve quadratic equations by factoring, completing the square, graphing and using the quadratic formula.

Students need to be able to solve equations involving polynomial, rational, and radical expressions and estimate their solutions from graphs.

Students need to understand the effect of simplifying radical or rational expressions on the solution set of equations such as $x^{\frac{2}{x}} = x$ for $x \neq 0$. (what happens when x=4?)

Students need to understand that any equation in x can be interpreted as the equation $f(x) = g(x)$ and to be able to interpret the solutions of the equation as the x-value(s) of the intersection point(s) of the graphs of $y = f(x)$ and $y = g(x)$. Students should be given the opportunity to use this understanding to guide their work as they use technology to graph each side as a function and then approximate the solutions.

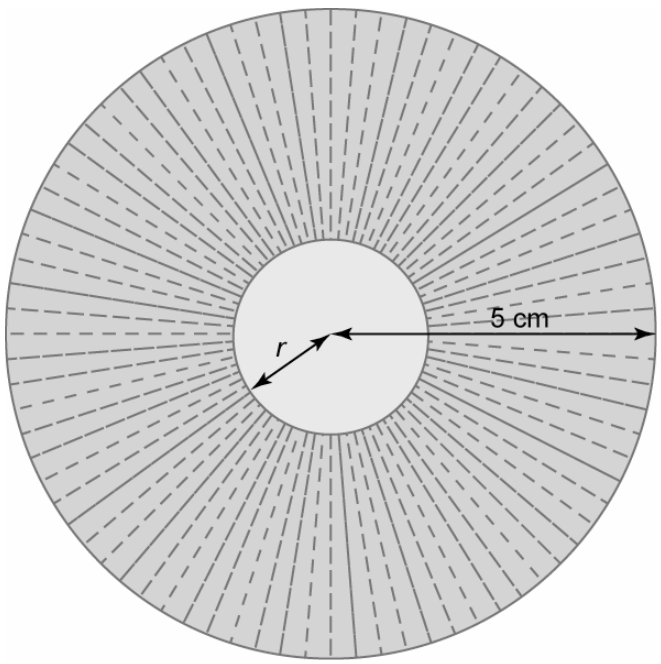
Students need to be able to solve the following systems of equations and inequalities and interpret their solutions graphically:

- Systems of linear equations in two and three variables
- Systems of linear and quadratic inequalities in two variables
- Systems of equations involving nonlinear expressions.

The following *Balanced Assessment* task is an example of a relevant problem for students in grades 11-12:

> A special machine, just like the one shown below, folds a flat circle of paper into cylindrical cupcake holders. The machine always starts with a flat circle of paper 5 centimeters in radius.

Circular piece of paper with folds indicated



> The special machine crimps the sides and then folds the flat circle of paper (radius 5 centimeters) into different sized cylindrical holders for cupcakes. These cupcake holders are special in that they are perfect cylinders.

1. The machine can be set to make different values for the radius of the base of the cupcake holder.
   Write a formula expressing the volume, V, of the cupcake holder in terms of the radius *r*. Show how it can be used to find the volume of a

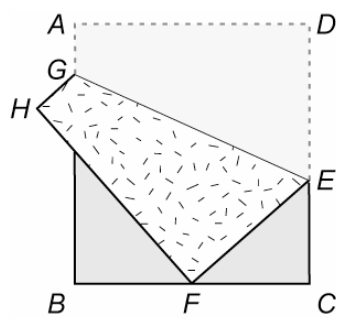
cupcake holder with a radius of your choice. (Remember, the machine always starts with a flat circle of paper 5 centimeters in radius.)

2. Find a good approximation for the radius *r* that gives the largest volume. Show how you arrived at your approximation and justify why you think this gives the largest volume.

The following task integrates aspects of geometric measurement and functions:

Folded Paper

The dimensions of a rectangular piece of paper *ABCD* are *AB* =10 and *BC* = 9. It is folded so that corner *D* is matched with a point *F* on edge *BC*. Given that length *DE* = 6, find *EF*, *EC*, and *FC*.



The area of triangle EFC is a function of the length DE. Write a formula for this function, using the letter x to stand for DE.

Find the value of x that maximizes the area of the triangle EFC.

Triangle Sides in Arithmetic Sequence Problem

Use the triangle inequality to show that a triangle can be formed with side lengths 2, 3, and 4.

Show that the sides of this triangle form an arithmetic sequence.

For what values of k can a triangle have sides whose lengths form an arithmetic sequence with common difference k? (Hint: Use the triangle inequality, letting the shortest side of the triangle have length *s*)

Triangle Sides in Geometric Sequence Problem

Use the triangle inequality to show that a triangle can be formed with side lengths:

$$1, \frac{5}{4}, \frac{25}{16}$$

Show that the sides of this triangle form a geometric sequence.

For what values of *k* can a triangle have sides whose lengths form a geometric sequence with common ratio, k? (Hint: Use the triangle inequality, letting the shortest side of the triangle have length *s*)

The greatest possible value of k that you found above has a special significance in mathematics. Do you recognize this number?

Grade Span 11 –12 for Prospective Mathematics Majors

In addition to the mathematics identified in the grade span 11-12, students who plan to major in mathematics, engineering, or the sciences need to study additional mathematical concepts.

Sequences

Prospective mathematics majors need the opportunity to compute partial sums of infinite arithmetic and geometric sequences, determine when an infinite geometric series converges, and find its sum. Students should also have the opportunity to understand arithmetic and geometric sequences as discrete forms of linear and exponential functions, respectively.

Functions and Relations

Prospective mathematics majors need the opportunity to understands functions and relations from a set-theoretic perspective, and be able to understand inverse and composition functions and compute inverses algebraically. Students should be able to understand domain restriction and its impact on the function and its properties.

Students should be able to analyze characteristics of classes of exponential functions, logarithmic functions, and trigonometric functions and their inverses. These analyses should include a study of domain, range, intercepts, increasing and decreasing intervals, rates of change, periodicity, end behavior, maximum and minimum values, continuity, and asymptotes. Students should be able to recognize properties of families of functions, including logarithmic and trigonometric, and graph them.

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Students should be able to analyze properties of functions including injectivity (1-1), surjectivity (onto), critical points and inflection points, and to determine graphically and analytically whether a function is even, odd, or neither. In addition, student should be able to analyze the idea of continuity and limits.

Expressions

Prospective mathematics majors should be able to use the remainder theorem, the factor theorem, and the rational root theorem for polynomials. Such students should understand the difference between factoring polynomials over integer, rational, real, and complex numbers.

Equations and Inequalities

Prospective mathematics majors need the opportunity to extend their study of equations by solving equations involving trigonometric, exponential, and logarithmic expressions algebraically before using graphing technology to check or approximate solutions.

Such students should extend their study of systems of equations by interpreting them as matrix equations and solving them with and without technology.

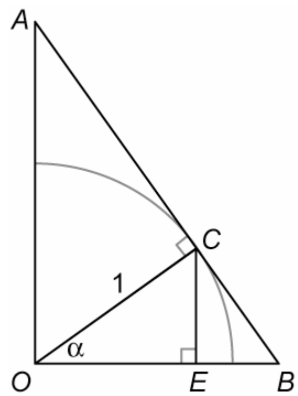
Students should also apply the intermediate value theorem to find exact or approximate solutions of the zeros of continuous functions.

Here are some examples of tasks that cut across various aspects of the algebra and functions outlined above:

Trigonometric Ratios

The quarter circle shown has radius OC = 1.

Represent each sin α, cos α, tan α, csc α, sec α, cot α as a length on the diagram.



Doubling Problem

A particular type of account earns interest at a rate r = 0.07 (= 7%) per year. (Is compounded annually needed anywhere here?)
That means that if you deposit $1,000 then after n years your account will be worth

$\$1{,}000\ (1.07)^n$.

In general, at an interest rate of r, after n years your $1,000 will be worth

$\$1{,}000\ (1 + r)^n$.

What interest rate r would be required if you want your $1,000 to *double* after 5 years?

In terms of *n*, say what interest rate *r* would be required if you want your $1,000 to *double* after *n* years.

In terms of m, say what interest rate r would be required if you want your $1,000 to *be multiplied by a factor of m* after 5 years.

In terms of m and n, say what interest rate r would be required if you want your $1,000 to *be multiplied by a factor of m* after *n* years.

Log/Log and Semi-Log Plots Problem

What sorts of functions $y = f(x)$ have straight line graphs when, instead of plotting (x,y), we plot $(x, \log_{10} y)$? Give examples. Generalize.

What sorts of functions $y = g(x)$ have straight line graphs when, instead of plotting (x,y), we plot $(\log_{10} x, \log_{10} y)$? Give examples. Generalize.

What sorts of functions $y = h(x)$ have straight line graphs when, instead of plotting (x,y), we plot $(\log_{10} x, y)$? Give examples. Generalize.

References:

Carpenter, T.P. Franke M. L, & Levi, L. (2003). *Thinking Mathematically, Integrating Arithmetic & Algebra in Elementary School*. Heinemann. Portsmouth, NH.

Balanced Assessment for the Mathematics Curriculum (1996) Dale Seymour.