Proportional Reasoning

A Research Based Unit of Study for Middle School Teachers

Rhode Island Department of Education
Office of Instruction
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Section 1: Purpose and Design

Proportionality is a keystone topic in middle school mathematics and receives a high degree of emphasis in the New England Common Assessment Program (NECAP) Grade-Level Expectations (GLEs). *Proportional Reasoning* is one unit of study in a series of units designed around the keystone topics in middle school mathematics.

The purpose of this unit of study is to support quality instruction by increasing teachers' mathematical content knowledge primarily through examining research findings related to students' understandings of proportional reasoning, along with examining student work. This unit of study will be particularly useful for teachers who are planning to teach a unit on proportionality. When adapting the ideas in these units of study to the classroom, teachers should purposefully design their lesson plans to incorporate three important principles of learning as identified by the National Research Council in *How Students Learn – Mathematics in the Classroom* and described in Table 1.1 on the following page (Donovan & Bransford, 2005). It should also be noted that this unit is not meant to supplant current curricula materials but rather to be used in conjunction with them. Furthermore, the three principles described in Table 1.1 are modeled throughout this unit of study. The Essential Questions in Section 3 seek to engage preconceptions, the various sections throughout the unit connect procedural knowledge to conceptual knowledge, and there are checkpoints along the way to monitor and reflect on your progress. Additionally, the summary section (Section 9) contains exercises allowing the participants to reflect upon their instructional programs and curricular materials.

Each unit of study is broken into sections that build upon one another. Furthermore, each unit of study begins by examining the Grade-Level Expectations that are pertinent to the particular unit of study. Following the identification of the expectations related to the unit of study, you will answer some essential questions. It is recommended that you answer these essential questions individually prior to reading subsequent sections. The essential questions will help frame the mathematical ideas of the unit. Exercises appear throughout the remaining sections. These exercises are imbedded within the section rather than at the end of the section and are intended to be solved and discussed as you are working through the section. Section 8 is devoted to examining NECAP released items and the student work that is available for these items. This section is subsequent to the sections that discuss research findings. So, once you reach this section, you will be able to make connections between the research and the NECAP items and identify typical student misconceptions.

This unit of study does not attempt to cover all areas of proportional reasoning, but rather is an introduction into the research behind the aspects of this topic that middle school students study. Ideas for content extensions and research extensions can be found by exploring the references in Section 11.

Additionally, it is worth noting that students need opportunities to work collaboratively, share ideas, and present ideas. Their understandings should be challenged and students should be allowed to build and construct knowledge for themselves. Students should actively engage in mathematics. Teachers should carefully guide this work, and should be cautious about just telling students the 'answers.' Teachers should ask students to explain how they know and allow them to share multiple ways to solve problems. Teachers need to continually probe students' understandings, especially by allowing students to explore new ideas on their own. Teachers need to resist modeling a few dozen low-level problems (i.e., not cognitively challenging) for students and then asking them to solve far too many homework problems all of which can be matched to one of the modeled problems.

Table 1.1 – Principles of Learning

Principle	Description
Principle 1 – Engaging Preconceptions	Students come to the mathematics classroom with ideas about the structures of mathematics and informal understandings. If their preconceptions are not engaged and if there is no bridge between informal and formal understanding, students may have difficulty learning new ideas and may continue to revert to their preconceived notions.
Principle 2 – Connecting Procedural/factual knowledge and Conceptual Understanding	Procedural knowledge and conceptual understanding must be balanced. When one places too much emphasis on procedural fluency the result is a lack of understanding in how the procedures work. Whereas, when one places too much emphasis on conceptual knowledge, often students lack the ability to perform the procedures in an efficient way. Teachers must help students build and connect ideas and organize knowledge into networks. It is important to discuss various solution methods and why they work and make connections among them.
Principle 3 – Self Monitoring	Students need to be afforded the opportunity to think about their own learning and assess their own mathematical progress. Eventually, such assessment opportunities will be internalized and students will begin to monitor their own progress.

Exercise 1.1 Take some time to complete the following graphic organizer in your journal. You should use this organizer throughout the unit of study and continually add to it. If you are working with colleagues, it is recommended that you each complete your own graphic organizer and post it on chart paper so that you can learn throughout the unit from the work of your colleagues.

Proportional reasoning is…	**Things that I'm unclear about regarding proportional relationships/What I want to learn…**
Students' misconceptions with proportional relationships include…	**Various ways to solve proportional problems include…**
Proportional relationships are…	**Things I learned include…**

Section 2: Connecting to the Grade-Level Expectations

Proportionality permeates the NECAP GLEs and represents a big idea in middle school mathematics. NCTM's Evaluation and Curriculum Standards states "[proportionality is] "of such great importance that is merits whatever time and effort must be expended to assure its careful development." (NCTM, 1989, p. 82) One of the goals of this unit of study is to help you become more familiar with the Grade-Level Expectations dealing with proportionality, including the research behind them. In order to achieve this goal, you will spend some time looking through the Grade-Level Expectations (see Exercise 2.1) to find the standards related to proportionality.

In Section 9 you will spend time analyzing released NECAP items and student work associated with these items. Please complete Exercise 2.1 before reading on.

Exercise 2.1 Locate the GLEs that deal with proportionality. Make certain that you are considering both state and local GLEs.

Please refer back to the *About the Mathematics GLE* page at the beginning of the GLEs if necessary to recall how bolding and underlining is used in the document.

Exercise 2.2 Locate the Curriculum Focal Points and Connections at elementary grades that deal with proportional reasoning.

While many GLEs are connected to proportional reasoning, including GLEs dealing with ratio and proportion, percents, similarity, scaling, linear equations, linear patterns and relationships, slope, rates, relative frequency histograms, and probability, this unit of study will primarily deal with content contained within the following GLEs (New Hampshire Department of Education & Rhode Island Department of Education, *NECAP and Local Mathematics Grade-Level Expectations for grades K-8*, 2005).

Grade 6

M:N&O:6:1 **Demonstrates conceptual understanding of rational numbers with respect to** ratios (comparison of two whole numbers by division a/b, $a : b$, and $a \div b$, where $b \neq 0$); and rates (e.g., a out of b, 25%) **using models, explanations, or other representations**.

M:G&M:6:5 **Demonstrates conceptual understanding of similarity** by describing the proportional effect on the linear dimensions of polygons or circles when scaling up or down while preserving the angles of polygons, or by solving related problems (including applying scales on maps). Describes effects using models or[sc] explanations.

M:F&A:6:1 **Identifies and extends to specific cases a variety of patterns** (linear and nonlinear) represented in models, tables, sequences, graphs, or in problem situations; or writes a rule in words or symbols for finding specific cases of a linear relationship; or writes a rule in words or[sc] symbols for finding specific cases of a nonlinear relationship; and writes an expression or[sc] equation using words or[sc] symbols to express the **generalization** of a linear relationship (e.g., twice the term number plus 1 or[sc] $2n + 1$).

M:F&A:6:2 **Demonstrates conceptual understanding of linear relationships** ($y = kx$; $y = mx + b$) **as a constant rate of change** by constructing or interpreting graphs of real occurrences and describing the slope of linear relationships (faster, slower, greater, or smaller) in a variety of problem situations; **and describes how change in the value of one variable relates to change in the value of a second variable** in problem situations with constant rates of change.

Grade 7

M:N&O:7:4 **Accurately solves problems involving** proportional reasoning; percents involving discounts, tax, or tips; and rates.

(IMPORTANT: *Applies the conventions of order of operations including parentheses, brackets, or exponents*.)

M:G&M:7:5 **Applies concepts of similarity** by solving problems involving scaling up or down and their impact on angle measures, linear dimensions and areas of polygons, and circles when the linear dimensions are multiplied by a constant factor. Describes effects using models or[sc] explanations.

M:F&A:7:1 **Identifies and extends to specific cases a variety of patterns** (linear and nonlinear) represented in models, tables, sequences, graphs, or in problem situations; **and generalizes** a linear relationship using words and symbols; generalizes a linear relationship to find a specific case; or writes an expression or[sc] equation using words or[sc] symbols to express the **generalization** of a nonlinear relationship.

M:F&A:7:2 **Demonstrates conceptual understanding of linear relationships** ($y = kx$; $y = mx + b$) **as a constant rate of change** by solving problems involving the relationship between slope and rate of change, by describing the meaning of slope in concrete situations, or informally determining the slope of a line from a table or graph; **and distinguishes between constant and varying rates** of change in concrete situations represented in tables or graphs; **or describes how change in the value of one variable relates to change in the value of a second variable** in problem situations with constant rates of change.

Grade 8

M:N&O:8:4 **Accurately solves problems involving** proportional reasoning (percent increase or decrease, interest rates, markups, or rates); multiplication or division of integers; and squares, cubes, and taking square or cube roots.

(IMPORTANT: *Applies the conventions of order of operations*.)

M:G&M:8:5 **Applies concepts of similarity** to determine the impact of scaling on the volume or surface area of three-dimensional figures when linear dimensions are multiplied by a constant factor; to determine the length of sides of similar triangles, or to solve problems involving growth and rate.

M:F&A:8:1 **Identifies and extends to specific cases a variety of patterns** (linear and nonlinear) represented in models, tables, sequences, graphs, or in problem situations; **and generalizes** a linear relationship (non-recursive explicit equation); generalizes a linear relationship to find a specific case; **generalizes** a nonlinear relationship using words or[sc] symbols; or **generalizes** a common nonlinear relationship to find a specific case.

M:F&A:8:2 **Demonstrates conceptual understanding of linear relationships** ($y = kx$; $y = mx + b$) **as a constant rate of change** by solving problems involving the relationship between slope and rate of change; informally and formally determining slopes and intercepts represented in graphs, tables, or problem situations; or describing the meaning of slope and intercept in context; **and distinguishes between linear relationships (constant rates of change) and nonlinear relationships (varying rates of change)** represented in tables, graphs, equations, or problem situations; or **describes how change in the value of one variable relates to change in the value of a second variable** in problem situations with constant and varying rates of change.

Section 3: Essential Questions

Essential questions help you to begin to think about the mathematics that will be the focus of this unit of study. You are encouraged to think deeply about these questions and to work them independently before discussing your thoughts with colleagues and before reading subsequent sections. It is also recommended that you keep a journal that contains your work on these problems and the problems throughout this unit of study. You are encouraged to use PEN in your journal so that you can not easily erase your work. Even though this is contrary to what many mathematics teachers ask students to do, using pen will allow you to go back and reflect on your initial thoughts, analyze any errors that you have made, and see how your learning has developed. (This is a suggestion that Tim Kurtz, NH State Assessment Director, has passed on to me that I try to share whenever possible. As a teacher, if you require your students to use pen, you will be able to easily identify what students were thinking when working problems and any errors made by students. This will facilitate your efforts in addressing students' preconceptions and misconceptions and will allow students to monitor their progress – See Table 1.1.) Some of these questions are intentionally vague in some areas. The reasons for the intentional vagueness will be apparent when one works through the remainder of the sections. Many of these questions will be discussed throughout various sections of this unit of study; therefore, please refrain from looking at the answers to the essential questions (Section 10) until working through the entire unit of study. Additional questions will be posed throughout the various sections.

Essential Question 1 Kris and Rich like to skate laps together around an ice rink since they both skate at the same constant rate. Today, Kris started skating first. By the time that Kris had completed 9 laps, Rich had completed 3 laps. How many laps will Kris complete by the time that Rich completes 15 laps? Explain. (Adapted from Lamon, 1999)

Essential Question 2 3 bags of mulch weigh 21 pounds. How many pounds does 8 bags of mulch weigh?

Essential Question 3 Alisa is painting her living room. She can paint the entire living room in 4 hours. Assuming that Karen can complete the job in the same amount of time as Alisa, how long will it take Karen and Alisa to paint the living room together? (Adapted from Lamon, 1999)

Essential Question 4 How many thirds are there in one-half? Draw a model to show how many thirds are in one-half.

Essential Question 5 If one player on a soccer team weighs 170 pounds, then what is the weight of 4 players?

Essential Question 6 Ray and Crystal invested money in a business and will split the profit in a ratio of 2:3. If the profit from the business is $1000, how much money will each person receive? Show how to use a model to solve the problem.

Essential Question 7 Can Courtney enlarge a 3 inch by 5 inch photograph proportionally to a 4 inch by 6 inch photograph?

Essential Question 8 Which relationships in questions 1–6 can be modeled by proportional relationship? Which are non-proportional relationships? Explain.

Essential Question 9 Describe how you determined whether the relationships in questions 1–6 were proportional or non-proportional. Identify any key words or reasoning that you used.

Essential Question 10 Determine some typical student responses to question 1.

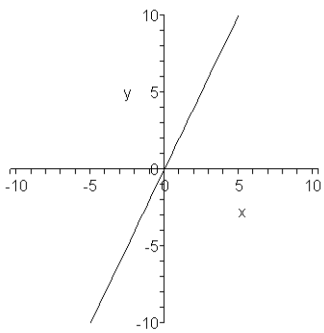
Essential Question 11 Describe which linear relationships are proportional and which linear relationships are non-proportional.

Essential Question 12 Describe various solution methods that students use when solving exercises involving proportionality.

Essential Question 13 Given two rectangles, one with length 38 inches and width 34 inches and one with length 28 inches and width 25 inches, which rectangle is more square? Describe what this problem has to do with proportionality.

Essential Question 14 Which of the following graphs represent proportional relationships? Which represent non-proportional relationships? Explain.

A



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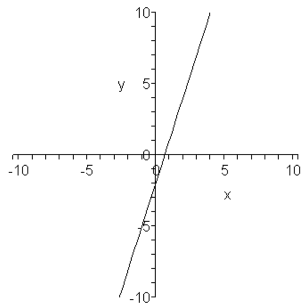

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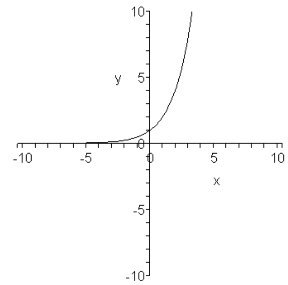

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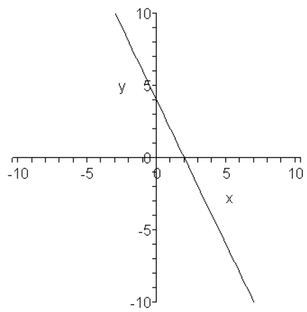

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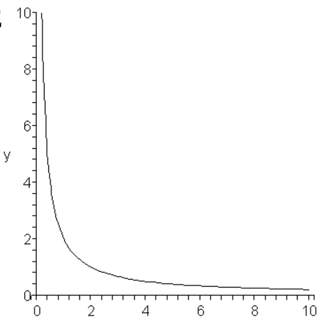
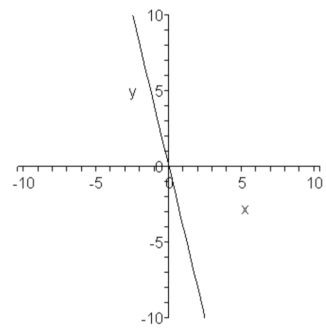


F



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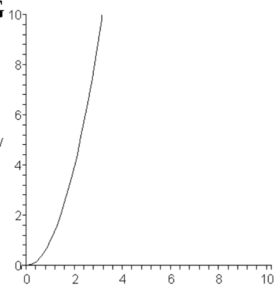

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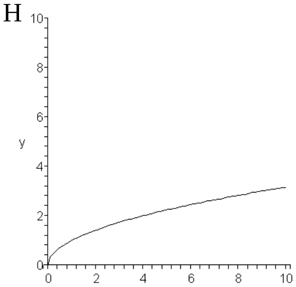
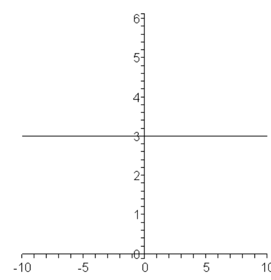


I



Graphs generated with Maple 9.5.

Essential Question 15 Describe how one variable changes in relation to another variable in directly proportional relationships.

As you can see from our essential questions, careful consideration needs to be made to determine whether or not a relationship is a proportional relationship. Determining that there are three known quantities and one unknown quantity in a problem is not sufficient to determine whether or not the relationship is a proportional relationship. Hence, the ability to set up a proportion and solve it does not constitute proportional reasoning. Additionally, it is evident that there are different types of problems that deal with proportional reasoning and various solution strategies. So what exactly is proportional reasoning?

Section 4: What is Proportional Reasoning?

You should have gained some insight into what is meant by proportional reasoning by working through the essential questions in Section 3. Hopefully you came to the conclusion that reasoning proportionally is much more than the ability to set up and solve an equation and entails being able to distinguish between proportional and non-proportional relationships. Before reading on, take some time to revisit your graphic organizer from Exercise 1.1 and think about what proportional reasoning is. Write your ideas in your journal. While it is difficult to completely describe what it means to reason proportionally, researches have begun to identify some of the critical components to proportional reasoning as shown in Figure 4.1. (Lamon, 1999)

This section will briefly describe these components and define what is meant by a proportional relationship (see Table 4.2). While this unit of study doesn't begin to try to fully discuss all of these components, you should be cognizant of them while reading subsequent sections and continually looking for how the examples and exercises connect to these various components. It should be noted that proportional reasoning is something that is built through experience and takes time. One can not simply teach a section on each of the following components.

Figure 4.1 – Components that Contribute to Proportional Reasoning

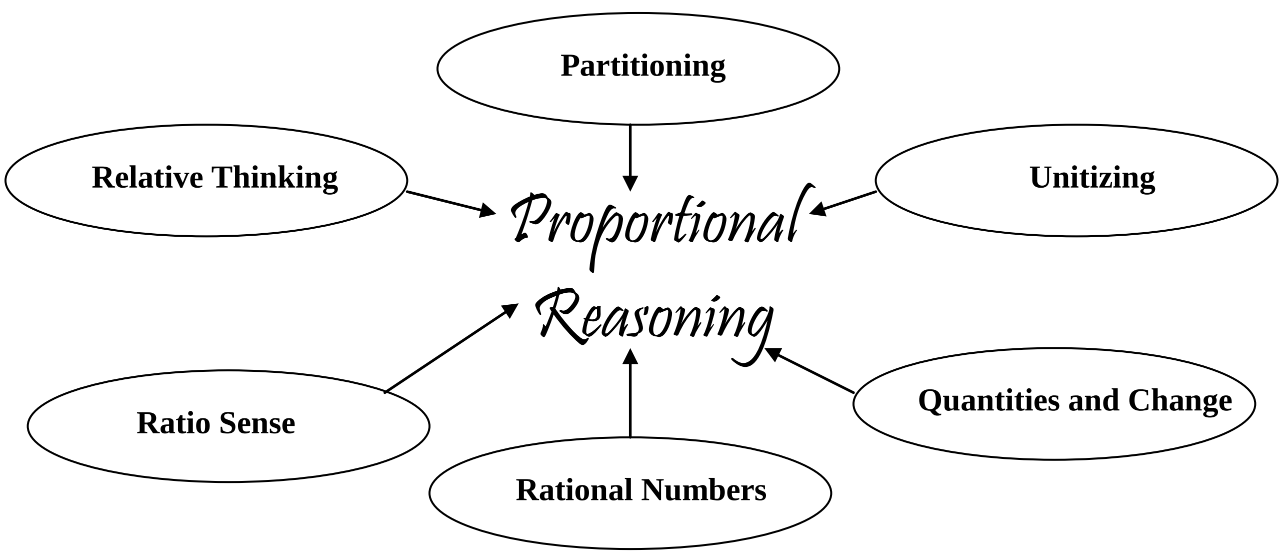


Figure 4.1 recreated from Teaching Fractions and Ratios for Understanding, copyright 1999 by Lawrence Erlbaum Associates, Inc. All rights reserved.

Table 4.1 – Description of Components Contributing to Proportional Reasoning (Adapted from Lamon, 1999)

Components of Proportional Reasoning	Description
Relative Thinking	Relative thinking can be described as thinking multiplicatively. It is important for students to understand both absolute change and relative change. Making the transition from absolute change to relative change is an important step in transitioning from additive to multiplicative reasoning. (See Example 5.1)
Partitioning	The partitioning of an object is the process of dividing the object into a number of disjoint parts that collectively make the whole. When dealing with fractions, to determine fractional parts, one partitions the object into parts of equal size.
Unitizing	Unitizing is the cognitive process used to assign a unit to a given quantity. For example, when asked to think about a case of soda, do you picture 24 cans, two 12-packs, or four 6-packs [Lamon, 2006]? It is desirable for students to build flexibility in determining the size of the chunk of a quantity that they use for a unit, as different situations may call for different sized chunks. Asking students to explain their choices can help encourage this flexibility.
Ratio Sense	Ratio Sense involves the ability to think flexibly in problem situations involving ratios.
Rational Numbers	Rational Numbers are numbers of the form $\frac{a}{b}$, where $b \neq 0$ and a and b are integers. That is, numbers that can be written as fractions. More importantly, reasoning with rational numbers requires the ability to reason flexibly with fractions, ratios, rates, and percents, and the operations on them.
Quantities and Change	Quantitative reasoning involves the ability to interpret and operate with change. This may require operating with constant or varying rates of change.

Table 4.2 – Definitions

Term	Definition
Ratio	A ratio is a quotient of two numbers or quantities. Ratios can compare similar units of measure or unlike units of measure (e.g., 100 miles per 2 hours). Ratios that compare unlike units are called rates. (NH Dept. of Ed., RI Dept. of Ed., & VT Dept. of Ed. NECAP Support Materials, 2004)
Proportion	A proportion is the equality of two ratios.
Directly Proportional (Direct Variation)	Two quantities, y and x, are said to be directly proportional if their ratio is some nonzero constant k. That is, $\frac{y}{x} = k$. Alternatively, this means that $y = kx$. The constant k is often referred to as the constant of proportionality or the quantity that is invariant, while y and x are said to be covariant.
Indirectly Proportional (Indirect Variation)	Two quantities, y and x, are said to be indirectly proportional if their product is some nonzero constant k. That is, $xy = k$. Alternatively, this means that $y = \frac{k}{x}$. The constant k is often referred to as the constant of proportionality or the quantity that is invariant, while y and x are said to be covariant.

This unit of study will focus mainly on directly proportional relationships. However, since it is important for students to see both directly and indirectly proportional relationships, we will spend some time looking at indirectly proportional relationships. It is important to note that often the word "directly" is left off when one is referring to directly proportional relationships. For example, if one asks whether or not a relationship is a proportional relationship, usually one is asking if the relationship is a directly proportional relationship (i.e., can the relationship be modeled by a function of the form $y = kx$); however, proportional relationships include both direct and indirect relationships.

Exercise 4.1 Revisit Essential Questions 8 and 14 in Section 3. Given that the questions didn't specify "directly" or "indirectly" proportional relationships, go back and rework these two exercises by classifying the relationships as directly proportional, indirectly

proportional, or neither. For the directly and indirectly proportional relationships identify which quantities are covariant and identify the invariant.
Notice that the constant of proportionality (in both direct and indirect situations) can be negative. While it is typical for middle school students to work in problem-solving situations where k is positive, along with both x and y (and this unit of study will focus mainly on these applications), it is important to note that this isn't always the case. Therefore, if we revisit Essential Question 14 in Section 2 we see that graphs A and F represent directly proportional situations (i.e., each is a line that passes through the origin and has the form $y = kx$ – note that k is less than zero for graph F) and graph E represents an indirectly proportional situation (i.e., can be modeled by $y = \frac{k}{x}$ – note graphs G and H pass through the origin which can never happen for an indirectly proportional situation since the product of x and y is always a nonzero constant; also, graph C passes through (0, 1) and (2, 4) so we can see that the product of x and y isn't a constant).

Exercise 4.2 Create a table of x and y values that represent an indirectly proportional situation where k is positive (also make certain to pick some negative values for x). Repeat for k negative. Sketch graphs for both of your situations.

Let's revisit Essential Question 15.

> **Essential Question 15** Describe how one variable changes in relation to another variable in directly proportional relationships.

We can see from graphing directly proportional relationships that they are lines that pass through the origin. Hence, they have a constant rate of change which is given by their slope. Directly proportional situations are often described as situations where as x increases y increases (as in graph A in Essential Question 14 of Section 3); whereas, indirectly proportional situations are often described as situations where as x increases y decreases (as in graph E in Essential Question 14 of Section 3). While this is true for most middle school applications some caution needs to be taken. That is, these descriptions are true when the constant of proportionality, k, is positive (take a moment to verify this). However, you will notice that in situations such as graph F in Essential Question 14 of Section 3, as x increases y decreases, but this is a directly proportional situation. One always needs to be careful regarding whether or not the constant of proportionality is positive or negative when making such assertions. The important point is that in directly proportional situations there is a constant rate of change. That is, for each unit increase in x there is a fixed increase (for k positive) or decrease (for k negative) in y (e.g., each time x increases by 1, y increases by 3 or each time x increases by 1, y decreases by 4). Take a moment to discuss this.

It is important for students to build flexibility in working between multiple representations (e.g., graphs, symbolic form, tables).

While completely describing proportional reasoning is difficult, it is clear that “solving problems involving proportional reasoning means to use proportional reasoning in problem solving situations that may involve ratios, proportions, rates, slope, scale, similarity, percents, probability, and others. It is assumed that throughout instruction students have sufficient opportunities to connect each of these situations to proportional reasoning” (NH Dept. of Ed., RI Dept. of Ed., & VT Dept. of Ed. NECAP Support Materials, 2004, p. 36).

Now that we have a better understanding of what proportional reasoning entails we will spend some time in the next section discussing how to build multiplicative reasoning in early grades.

Section 5: Building Multiplicative Reasoning in Early Grades

The multiplicative reasoning that students begin to develop in early grades is fundamental to the study of proportionality in middle school. Thus, it is important to identify opportunities that allow children to reason multiplicatively in elementary school, and create exercises and tasks that challenge students' understandings of additive reasoning. Furthermore, NCTM calls for a greater emphasis on multiplicative reasoning in grades 3-5 (NCTM, 2000). While the focus of this unit of study is on proportional reasoning at middle school, it is important to understand the types of experiences that will provide students with the opportunity to begin to think proportionally in early grades. "The operating theory for instruction is that by providing children experiences with some of the critical components of proportional reasoning before proceeding to more abstract, formal presentations, we increase their chances of developing proportional reasoning" (Lamon, 1999, p. 3). Before reading on, take some time to think about ways that you might encourage proportional reasoning in early grades. Write your ideas in your journal.

The ability to analyze change in a relative sense, rather than just an absolute sense, is an important building block to proportional reasoning. Absolute thinking is additive thinking and relative thinking is multiplicative thinking. Example 5.1 illustrates the difference between absolute and relative reasoning.

Example 5.1 – Illustrating Absolute vs. Relative Change

For a science experiment, Quinn planted some sunflower seeds in two pots and began to make some observations. She controlled the amount of light that each plant received while making certain to hold other important variables constant (e.g., the amount of water that each plant receives, the amount of nutrients that each plant receives, the soil conditions) in an effort to determine how the amount of sunlight each plant receives affects the height of each plant after various weeks. Here are some of Quinn's observations.

	Plant A	**Plant B**
Week 3	5 inches	8 inches
Week 4	7 inches	10 inches

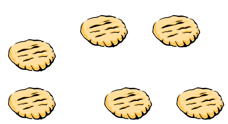
Which plant grew more between weeks 3 and 4? Take a moment to answer this question before reading on. Here are two approaches to answering the question.

Absolute Reasoning	Relative Reasoning
The plants grew the same about, 2 inches, since 7 – 5 = 2 and 10 – 8 = 2.	Plant A grew more. Plant A grew 2 inches between weeks 3 and 4 which is 2/5 of its height in week 3. Plant B also grew 2 inches but this is 2/8 of its height in week 3 and 2/5 is greater than 2/8 (since the numerators are the same and the denominator is larger in 2/8, making the parts smaller).

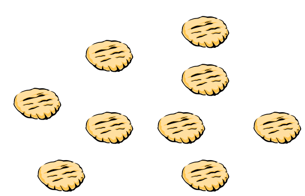
Notice the question in Example 5.1 doesn't specify which approach one should take when solving the problem. When students are given the opportunity to discuss their approaches to problems like Example 5.1, often they will find more than one way of looking at the problem. In general, providing students with well crafted, intentionally vague questions can help students build new knowledge and discover ideas such as relative thinking. One can also purposely ask questions that require absolute thinking or relative thinking. Providing students with both experiences will help students build the flexibility that they need to become effective problem solvers. Example 5.2 illustrates some questions requiring absolute thinking and some questions requiring relative thinking for the same problem situation.

Example 5.2 – Keeping the Problem Situation the Same, Varying the Types of Questions (Adapted from Lamon, 1999)

Look at the number of cookies that Marcus has and the number of cookies that Nadia has.



Marcus' cookies



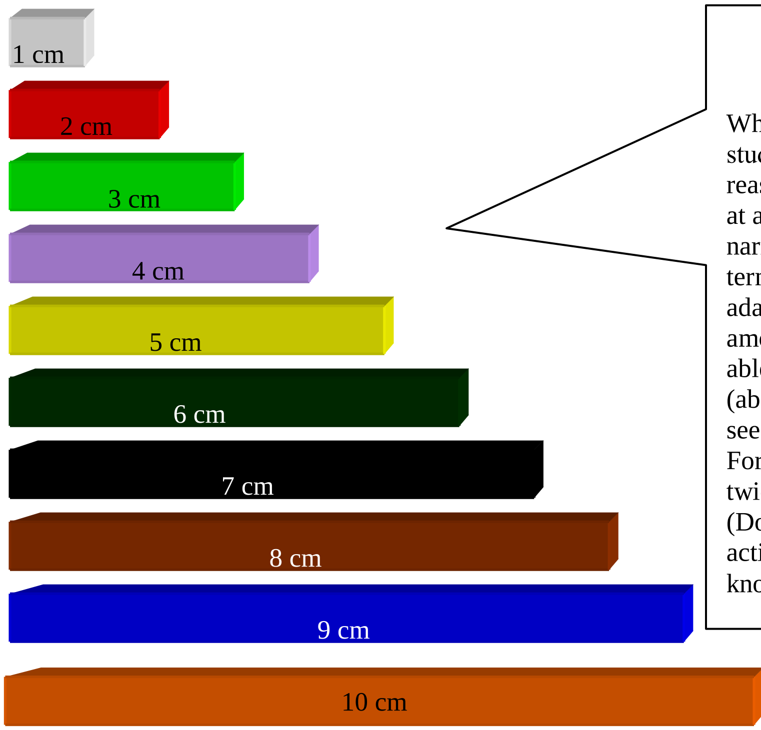
Nadia's cookies

Question Types

Absolute Thinking	Relative Thinking
Who has more cookies, Marcus or Nadia?	How many times would you have to stack up Marcus' cookies to get a pile as high a Nadia's cookies?
How many more cookies does Nadia have than Marcus?	What fraction of a dozen cookies does Nadia have?
How many fewer cookies does Marcus have than Nadia?	There are 6 cookies in a package of cookies. What part of a package of cookies does Marcus have? Nadia have?

Exercise 5.1 – Using Cuisenaire® Rods to Assess Relative and Absolute Thinking

Create some questions that can be used to assess relative and absolute thinking that make use of Cuisenaire® rods. Create some questions that are intentionally vague and some questions that are purposely designed to assess either relative or absolute thinking. Rods may be directly compared or used as a measuring tool. Here is a reminder of the lengths represented by the different colors of most Cuisenaire® rods.



Engaging Students Preconceptions
(See Table 1.1)

While it takes quite a bit of time to develop students' formal understandings of proportional reasoning, researchers have shown that students at a young age have little difficulty perceiving narrow upright containers in proportional terms. The idea in this exercise may easily be adapted by filling containers with various amounts of liquid. Not only will the students be able to see which containers have more liquid (absolute reasoning), but they will be able to see the amounts of liquid in proportional terms. For example, use terms such as half as much, twice as much, or one-quarter as much (Donovan & Bransford, 2005). These types of activities will help students build new knowledge from their existing preconceptions.

In addition to asking questions about relative and absolute differences, teachers can provide students in early grades with tasks similar to those that appear in middle school (see Exercise 5.2). Providing exercises that involve models may help students visualize the situation. While many students in early grades may have difficulty solving these types of problems, providing them with these experiences and giving them the opportunity to discuss their solution strategies can help make the transition from additive thinking to multiplicative thinking. We close this section with Exercise 5.2 which explores a released NAEP (National Assessment of Educational Progress) item and some of the various solution strategies that students use when approaching the problem, including strategies that are transitional from additive to multiplicative reasoning. Section 7 will further examine tasks to assess proportional reasoning along with various solution strategies and levels of understanding. The next section will discuss distinguishing between proportional and non-proportional relationships.

Exercise 5.2 – Modified NAEP 1996 Released item (As shown in Kenney, Lindquist, & Heffernan, 2002)

A fourth grade class needs 5 leaves each day to feed its 2 caterpillars.
How many leaves would they need each day for 12 caterpillars?

Answer:____

Use drawings, words, or numbers to show how you got your answer.

a) Explain how students might solve this problem correctly by counting by 2's and 5's.

b) Explain how students might incorrectly use additive reasoning to solve this problem resulting in an answer of 15 leaves.

c) Explain how students might solve this problem correctly by finding the number of leaves needed for 1 caterpillar.

d) Suppose Beth answers 15. What type of questions could you ask Beth to help guide her to the correct answer?

Section 6: Distinguishing between Proportional and Non-Proportional Relationships

The ability to distinguish which types of relationships are proportional and which types of relationships are non-proportional is a key aspect of proportional reasoning. Students need to be exposed to both proportional and non-proportional situations so that they can learn to determine when it is appropriate to use a multiplicative solution strategy (Cramer, Post, & Currier, 1993). Without these experiences, particularly in missing value problems (i.e., problems involving three known quantities and one unknown quantity), students tend to inappropriately model non-proportional problem situations with proportions as described in the following examples.

Example 6.1 – Revisiting Essential Questions 1 and 2 from Section 3

Thirty-three pre-service elementary education teachers taking a mathematics methods class and studying a unit on ratios and proportions were given two problems similar to Essential Questions 1 and 2 from Section 3 (these essential questions only differ in context from those originally given in the research study). Thirty-two of the thirty-three pre-service teachers solved the problem analogous to Essential Question 1 incorrectly, while all of them solved the problem analogous to Essential Question 2 correctly. However, none of them were able to explain why only one of the questions represented a proportional situation (Cramer, Post, & Currier, 1993). Let's take a closer look at these two Essential Questions.

> **Essential Question 1** Kris and Rich like to skate laps together around an ice rink since they both skate at the same constant rate. Today, Kris started skating first. By the time that Kris had completed 9 laps Rich had completed 3 laps. How many laps will Kris complete by the time that Rich completes 15 laps? Explain. (Adapted from Lamon, 1999)
>
> **Essential Question 2** If 3 bags of mulch weigh 21 pounds, how many pounds will 8 bags of mulch weigh?

The temptation in Essential Question 1 is to set up a proportion to solve the problem, since there are three known quantities and one unknown quantity, resulting in 45 for the answer as shown below.

$$\frac{9}{3} = \frac{x}{15}$$
$$9 \cdot 15 = 3 \cdot x$$
$$45 = x$$

The thinking behind this approach is that 9 laps for Kris is to 3 laps for Rich as x laps for Kris is to 15 laps for Rich. However, this particular problem can not be represented by setting up a proportion, so this approach is incorrect.

To see that the above approach will not work, reflect on whether or not 45 laps is a reasonable answer. Take a moment to think about this before reading on.

If Kris and Rich are skating at the same constant speed then each time Kris completes one more lap, Rich also completes one more lap. Since Kris started 6 laps ahead of Rich, he should end up completing 6 more laps than Rich or a total of 15 + 6 = 21 laps.

Let's take a closer look at this analysis of the situation by making a table to represent the information in the problem.

Number of laps Rich Completes	Number of laps Kris completes
3	9
4	10
5	11
6	12
7	13
8	14
9	15
10	16
11	17
12	18
13	19
14	20
15	21

The correct answer, 21 laps, can be seen by making a table.

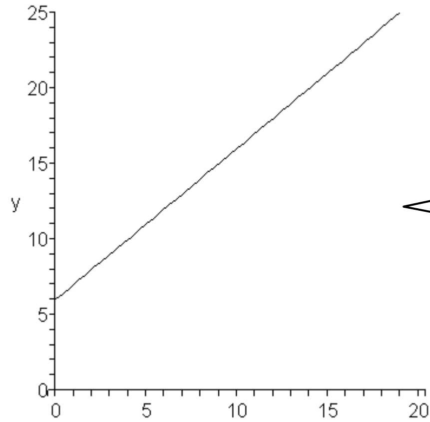
While we can see that making a table results in the correct answer and setting up a proportion results in an unreasonable answer, we still want to dig deeper into why a proportion can not be used to model this situation. Complete Exercise 6.1 before reading on.

Exercise 6.1 Why can't the "Skating Laps" problem be modeled by a proportion?

To help answer Exercise 6.1 it would be useful to extend the table in the other direction as well.

Number of laps Rich Completes	Number of laps Kris completes
0	6
1	7
2	8
3	9

Looking at the tables it becomes clear that the total number of laps Kris has completed is six more than the total number of laps that Rich has completed. If y represents the total number of laps Kris completes and x represents the total number of laps Rich completes, then the situation can be modeled by the linear function $y = 1x + 6$. Hence, the function has an additive component and isn't of the form $y = kx$, for k a constant, and therefore doesn't represent a directly proportional relationship. Notice that graphing this function produces a line with a y-intercept at (0, 6) since Kris started 6 laps ahead of Rich. Recall that proportional relationships are modeled by lines that pass through the origin. Additionally, we notice that the number of laps Kris completes divided by the number of laps Rich completes isn't a constant (i.e., $\frac{y}{x} \neq k$).



Graph representing the "Skating Laps" problem – non-proportional situation, where y represents the number of laps Kris completed and x represents the number of laps Rich completed

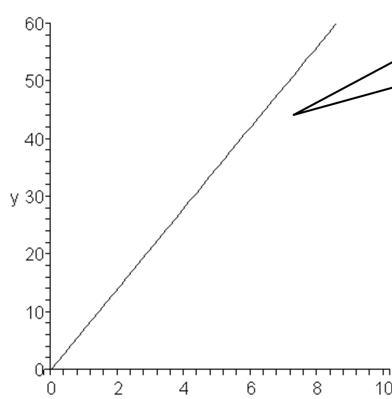
Graph generated in Maple 9.5

Exercise 6.2 What does the slope of the graph represent in the above situation?

In Essential Question 2, it is implied that we are dealing with the same size bags of mulch. Since 3 bags of mulch weigh 21 pounds, 1 bag of mulch weighs 7 pounds. Hence, when determining the weight of 8 bags of mulch ones needs to only multiply the number of pounds per bag by the number of bags (i.e., 7 lb/bag × 8 bags = 56 lb). Thus, the situation can be modeled by the linear function $y = 7x$, where y represents the total weight of x bags.

Graph representing Essential Question 2 – proportional situation, where y represents the total weight of x bags

The number of pounds per bag remains constant (i.e., 21 lb/ 3 bags) AND the graph passes through the origin (i.e., 0 bags weigh 0 lb). Hence, $\frac{y \text{ lb}}{x \text{ bags}} = \frac{21 \text{lb}}{3 \text{bags}}$, or in this particular case $\frac{y}{8} = \frac{21}{3}$.



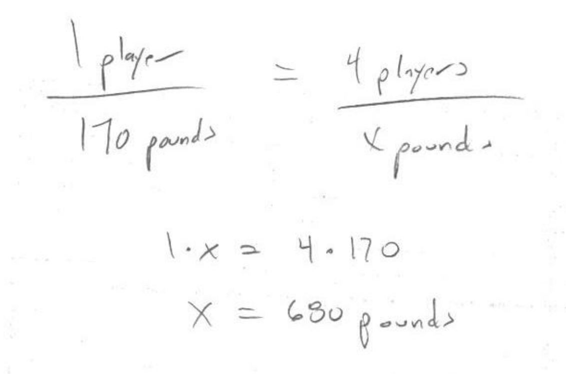
Exercise 6.3 Does knowing that a problem involves a constant rate guarantee that it can be represented by a proportional relationship? Explain. You may want to go back and take a moment to revisit Essential Question 9 from Section 3.

Example 6.2 – Revisiting Essential Questions 3 and 5 from Section 3

Take a moment to review Essential Questions 3 and 5 from Section 3 shown below.

> **Essential Question 3** Alisa is painting her living room. She can paint the entire living room in 4 hours. Assuming that Karen can complete the job in the same amount of time as Alisa, how long will it take Karen and Alisa to paint the living room together?
>
> **Essential Question 5** If one player on the soccer team weights 170 pounds, then what is the weight of 4 players? (Adapted from Lamon, 1999)

We will start by looking at Essential Question 5. A common solution to this problem is shown below.



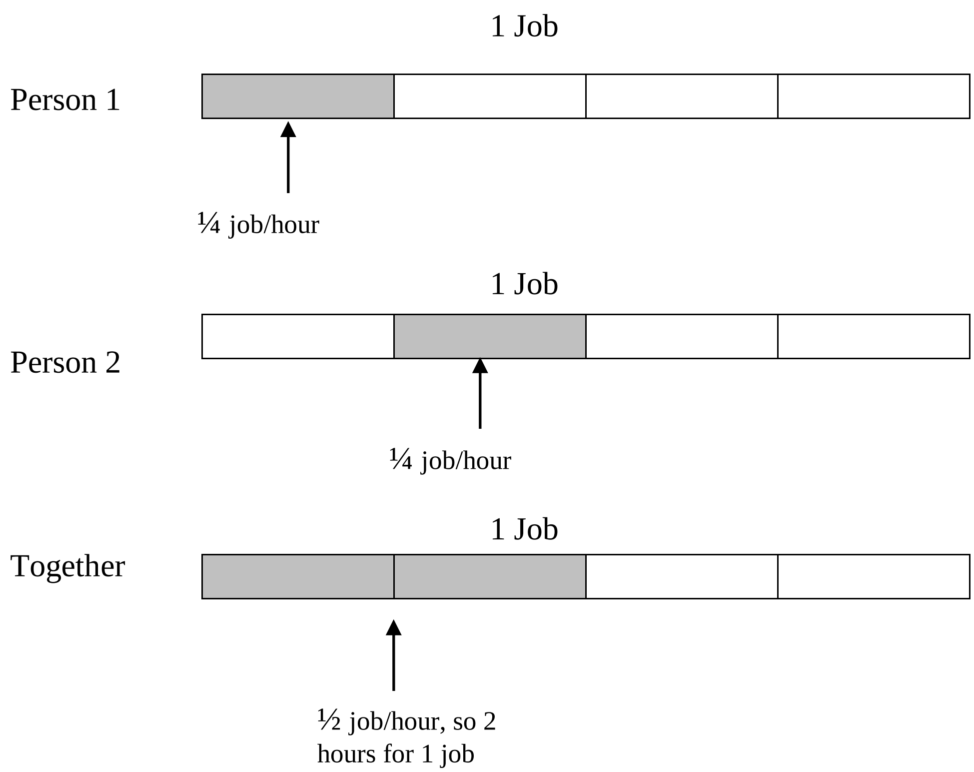
Again, since this situation involves three known quantities and one unknown quantity it is tempting to set up a proportion and solve the problem like shown above. However, it will rarely be the case that the weight of four players is four times the weight of a single player. That is, there is not a constant weight per player. To determine the weight of all four players one would need to add all four of their weights.

Setting up a proportion for Essential Question 3 is also tempting since it involves three known quantities and one unknown quantity. A common error is to set up the following proportion.

$$\frac{1\text{ person}}{4\text{ hours}} = \frac{2\text{ people}}{?\text{ hours}}$$

In this case, since the unknown is in the denominator, it is common for students to solve by using an equivalent fraction strategy (i.e., 1/4 is equivalent to 2/8).

Stopping to evaluate whether or not 8 hours is a reasonable answer quickly leads one to conclude that this problem can not be solved by setting up a proportion. To see the correct answer, one notices that both people can complete the job alone in 4 hours so it should take half as much time for the two people to complete the job together, or 2 hours. Another way to analyze the problem is to use a unit rate strategy – each person can complete 1 job every 4 hours so let's assume that each person can complete ¼ of a job per hour.



> Just because each person completes one job in the same amount of time, this doesn't imply that each person works at a constant pace or the same pace (e.g., the initial phase of the job may take considerable time, whereas the final phases go quicker), but making this assumption helps us model the situation. The important piece is that it takes 4 person-hours to complete one job and two people working together contribute two person-hours in one hour or four person-hours in two hours.

What happens when the number of people working on the job changes (still assuming that each person can complete one job alone in 4 hours)? Take a moment to think about this question before reading on.

Continuing with the above reasoning we realize that it should take 3 people one-third of the amount of time it takes a person to complete the job alone, and 4 people one-fourth of the time it takes a person to complete the job alone, 5 people one-fifth of the time it takes a person to complete the job alone, and so on. Thus, if y represents total amount of time for x people to complete one job, we have

$$y = \frac{4}{x}.$$

Equivalently, $xy = 4$ or (number of people working) × (total number of hours) = time for one person to complete the job.

Some caution needs to be taken when analyzing units as shown below.

$$y = \frac{4}{x}$$

This does not represent 4 hours/person, but rather 4 person-hours/job.

Organizing the information in a table can also help us analyze the situation to see that $xy = 4$.

Number of people (x)	**Total Time to Complete one Job in Hours (y)**	**Constant Product (xy)**
1	4	4
2	2	4
3	4/3	4
4	1	4
5	4/5	4
6	2/3	4

Using various representations (e.g., models, tables, symbolic form) can help us understand a situation more deeply. Essential Question 3 is an example of an indirectly proportional situation, or a situation that can be modeled by $y = \frac{k}{x}$, where k is a nonzero constant.

Exercise 6.4 Consider the following situation.

If it takes 6 construction workers 4 days to complete a job, how long will it take 2 construction workers to complete the job? [Assume that each construction worker can complete the entire job in the same amount of time when working alone.]

a) Explain what is wrong with the following reasoning.

> It takes 4 days per 6 people to complete the job, or it takes 2/3 day per person to complete the job times 2 people is 4/3 days or 1 1/3 days.

b) How long does it take one person to complete the job? Explain.

c) How many person-days does it take to complete one job?

d) Solve the original problem. Explain your method.

e) Model the situation with a rule that relates the total number of days, y, for x people to complete the job.

f) Create a graph to represent your rule in part e and connect the graph to multiplication facts.

In order for students to develop a deep understanding of proportional relationships it is important for them to be able to determine when a proportion is appropriate for a given situation. Too often curricula materials only address a single method of solution (cross multiplication) and do not involve problems in the same section/unit (or even chapter) that require students to distinguish between both proportional and non-proportional situations. Furthermore, these units often do not make geometric connections (i.e., directly proportional situations can be modeled by lines that pass through the origin) or connections to various other representations (e.g., tables, models). The next section discusses various types of tasks to assess proportionality along with various solution methods. This section closes with a few exercises.

Exercise 6.5 Think about whether or not your district curriculum and curricular materials provide students the opportunity to distinguish between situations that are proportional and situations that are not proportional along with the opportunity to work with various representations. You will revisit this exercise in Section 9 and specifically examine your materials when working through the exercises in Section 9.

Exercise 6.6 Create some problems that represent both proportional situations and non-proportional situations to try with your students. You should create examples of problems that can be modeled by linear functions that pass through the origin (i.e., of the form $y = mx$) and ones that have a y-intercept other than (0, 0) (i.e., of the form $y = mx + b$, where $b \neq 0$), along with problems that represent indirectly proportional situations (i.e., can be modeled by functions of the form $y = \frac{k}{x}$, where k is a nonzero constant).

> Students who develop a deep conceptual understanding of variables learn to be flexible with their use. Notice how in some cases we use k for the constant of proportionality (invariant) and other times, such as in directly proportional situations, we might use m to stress (in this case) that the constant of proportionality is also the slope of the line which models the given situation.

Exercise 6.7 Distance equals rate times time, or $d = rt$, is a fairly common model used throughout middle school, where r represents a constant rate. This model provides a familiar context to students and the opportunity to work with a directly proportional relationship, as in the example below.

> It takes Paul 2 hours to hike 5 miles, how long will it take him to hike 8 miles? 10 miles?

a) Identify the invariant in the above situation, along with the quantities that are covariant and solve the problem.

Intuitively, students understand that as the amount of time increases the amount of distance traveled also increases. This intuition can be used to introduce students to a similar situation that can be modeled by an indirectly proportional relationship as in the example below. Providing students with problems in familiar contexts, such as $d = rt$, and varying which quantity remains constant can help students build flexibility with proportional reasoning.

> It took Paul 8 hours to complete a kayak tour paddling at a pace of 3 miles per hour. Next week, Paul wants to complete the same tour in 6 hours. How fast should Paul paddle? How fast should Paul paddle if he wants to complete the tour in 4 hours?

b) Identify the invariant in the above situation, along with the quantities that are covariant and solve the problem.

Exercise 6.8 Ron was asked to look at the pattern in the table below and determine the Output when the Input is 8, and explain how he determined his answer.

Input	1	2	3	4	5	…	8
Output	7	12	17	22	27	…	?

Below is Ron's work.

Ron's work

> *Input: 8*
> *Output: ?*
>
> *When the Input is 4 the Output is 22.*
>
> $4 \times 2 = 8$
>
> $22 \times 2 = 44$
>
> *The Output is 44 when the Input is 8.*

a) By extending the pattern in the table we can see that Ron's response is incorrect and that the correct response is 42. Explain what is wrong with Ron's reasoning.

b) Identify a rule to determine the output when the input is *n* and explain, in at least two different ways, why the relationship shown is not a directly proportional relationship.

Section 7: Types of Assessment Tasks and Solution Strategies

This section will begin by examining different types of tasks that can be used to assess proportional reasoning and subsequently examine solution strategies to these various tasks. The goal is to provide ideas for research-based tasks that can be used for instruction and assessment and provide information regarding the difficulty of the tasks and various factors that influence this difficulty such as the context and numerical complexity.

The Rational Number Project (RNP) is an on-going research project investigating student learning and teacher enhancement that has been funded by the National Science Foundation (NSF) since 1979. As of the writing of this document the project has produced 86 research publications (http://education.umn.edu/rationalnumberproject/). A primary focus of the project is enhancing students' abilities in proportional reasoning and disseminating research findings regarding the teaching and learning of proportional reasoning that have the potential to impact classroom instruction.

The RNP identified four types of tasks that can be used to assess proportional reasoning:

- Missing Value Problems,
- Numerical Comparison Problems,
- Qualitative Prediction Problems; and
- Qualitative Comparison Problem.

Missing Value problems (see Example 7.1) involve three known quantities and one unknown quantity. In Numerical Comparison problems, two ratios are given and those ratios are to be compared (see Example 7.2). Qualitative Prediction and Comparison problems (see Examples 7.3 and 7.4) involve the use of proportional reasoning but do not depend upon numerical comparisons. Researches have determined that this type of reasoning is not the same as the type of reasoning used in solving missing value problems or numerical comparison problems (Heller et al., 1990 as cited in Cramer, Post 1993). Since the type of reasoning used in qualitative prediction and comparison problems is not dependent upon numerical comparisons, it is often seen as an important precursory step to calculations that allows students to analyze the reasonableness of their answers.

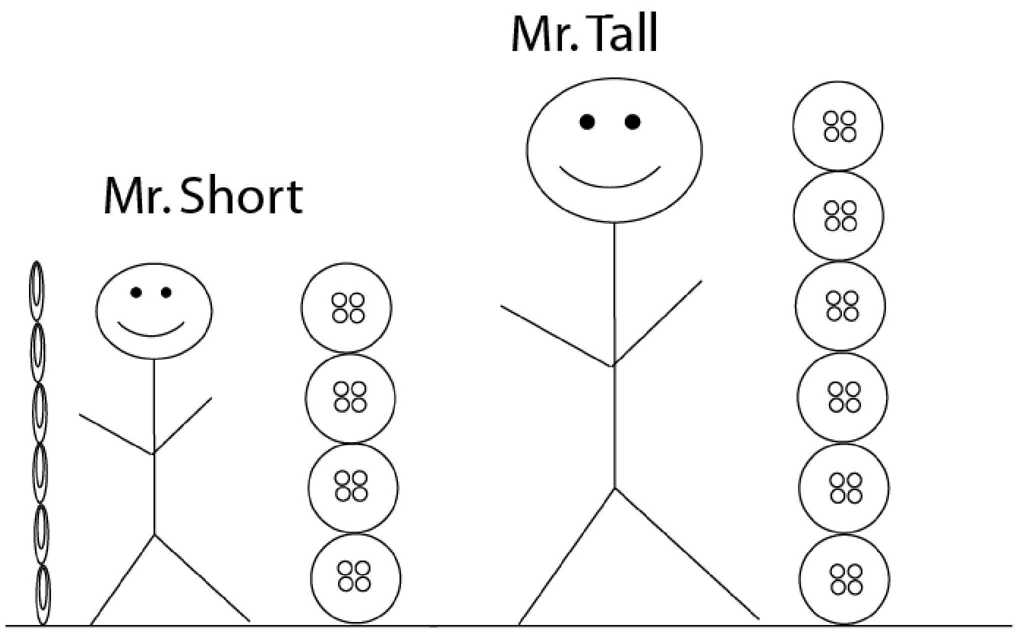
Example 7.1 – Missing Value Problems

Measurement Problem (Adapted from Karplus, Karplus, and Wollman, 1974 as cited in Khoury, 2002)

The length of Mr. Short is 4 large buttons.
The length of Mr. Tall is 6 large buttons.

When paper clips are used for measuring, the length of Mr. Short is 6 paper clips.

What is the length of Mr. Tall in paper clips? Explain.



Speed Problem Mike and Sam are running laps at the same pace around a track. It took Mike 8 minutes to complete 5 laps. How long did it take Sam to complete 7 laps?

Example 7.2 – Numerical Comparison problem

Mixture Problem Melanie and Louise are making lemonade. Melanie mixes 8 tablespoons of lemon mix with 64 ounces of water. Louise mixes 12 tablespoons of lemon mix with 128 ounces of water. Whose pitcher has the strongest tasting lemonade? Explain.

Notice that unlike the missing value problems in Example 7.1 that involve a constant ratio (i.e., in Mr. Short/Mr. Tall there are 6 paper clips per 4 buttons; in the speed problem we have 8 minutes per 5 laps) and can be modeled by a function of the form $y = mx$ (i.e., in Mr. Short/Mr. Tall if *y* represents the length in paper clips and *x* represents the length in buttons, then $y = \frac{3}{2}x$; in the speed problem if *y* represents time in minutes and *x* represents number of laps, then $y = \frac{8}{5}x$), the mixture problem in Example 7.2 involves proportional reasoning in the sense that it is using a ratio as a measure (i.e., comparing two rates). While one could model, for example, Melanie's mixture with a function of the form $y = mx$ (i.e., if *y* represents the number of tablespoons of lemon mix and *x* represents the number of ounces of water, then $y = \frac{1}{8}x$) and ask how many tablespoons of lemon mix she should use with 128 ounces of water (i.e., $y = \frac{1}{8} \cdot 128 = 16$) and conclude that for her mixture she would need 16 tablespoons of lemon mix, and thus Louise's mixture has a weaker lemon taste, this type of reasoning is not always as intuitive to students as comparing two rates. Discuss some problem situations when this type of reasoning is useful.

Various types of solution strategies to tasks involving proportional reasoning are discussed later in this section. We will also learn that limiting students' solution methods can be counterproductive to their development of a strong conceptual foundation.

Exercise 7.1 Consider the following problem (Essential Question 13 from Section 3):

- Given two rectangles, one with length 38 inches and width 34 inches and one with length 28 inches and width 25 inches, which rectangle is more square?

a) Solve the problem by using ratios as measures.

b) Solve the problem by modeling the situation with a function of the form $y = mx$ (similar to the above discussion) by using the first rectangle to determine the invariant and determining, based on this invariant, what the length of the rectangle would need to be if the width were changed to 25 inches. Determine how this helps you answer the original question.

c) Which method seems more intuitive? Explain.

Exercise 7.2 In Mrs. Keeley's class there are 5 boys for every 6 girls. Mrs. Keeley is conducting a probability experiment that requires her to randomly choose 11 students from her class.

a) Will Mrs. Keeley's sample contain 5 boys and 6 girls? Explain. (Adapted from Lamon, 1999)

b) Discuss the connection of this problem to proportional reasoning and using ratios as measures.

Example 7.3 – Qualitative Prediction problem

Mixture Problem If Melanie uses four more tablespoons of lemon mix today than what she used yesterday to make the same amount of lemonade, her lemonade today would have

a) a stronger tasting lemon flavor.
b) a weaker tasting lemon flavor.
c) a mix that has the same strength of lemon taste as yesterday.

Example 7.4 – Qualitative Comparison problem

Density Problem Alisa and Crystal are planting flowers in rows. Alisa planted more flowers in her first row than what Crystal planted in her first row. Alisa's first row of flowers is not as long as Crystal's first row of flowers. Whose flowers are planted closer together? Explain.

Exercise 7.3 Solve the problems in Example 7.1.

Exercise 7.4 Solve the problem in Example 7.2.

Exercise 7.5 Solve the problem in Example 7.3.

Exercise 7.6 Solve the problem in Example 7.4.

Exercise 7.7 – Qualitative Prediction/Comparison (adapted from Billings, 2002)

In each of the following questions the pictures represent two different glasses containing lemonade. The shaded portion of the glasses represents the amount of lemonade in the glasses.

a) Glass A contains lemonade that is stronger tasting than the lemonade in Glass B. If one teaspoon of lemon mix is added to Glass A and 3 ounces of water is added to Glass B, which glass contains the lemonade with the stronger lemon flavor? Explain.

Glass A Glass B

b) Glass A and Glass B contain lemonade that tastes the same (i.e., one does not have a stronger lemon flavor than the other). If one teaspoon of lemon mix is added to both Glass A and Glass B, which glass contains the lemonade with the stronger lemon flavor? Explain.

Glass A Glass B

c) Glass A contains lemonade with a weaker tasting lemon flavor than Glass B. If one teaspoon of lemon mix is added to both Glass A and Glass B, which glass contains the lemonade with the stronger lemon flavor? Explain.

Glass A Glass B

d) Glass B contains lemonade with a stronger tasting lemon flavor than Glass A. If one teaspoon of lemon mix is added to Glass A and one ounce of water is added to Glass B, which glass contains the lemonade with the stronger lemon flavor? Support your argument with examples.

Glass A Glass B

The RNP found that the context of proportional problems affected the difficulty of the problem even when the numerical values were held constant. Scaling was more difficult for middle grade students than other contexts (Cramer, Post, 1993). Additionally, researches have found that the type of numerical relationships used in the problem can affect how students interact with the problems. Furthermore, the difficulty students have with various numerical relationships can be due to misconceptions developed in early grades. For example, students often see fractions as two whole numbers rather than representing a single quantity. This can lead students to respond that there is no solution to missing value problems that have non-integral relationships as in Table 7.1 (Cramer, Post, & Currier, 1993). In order for students to become flexible with proportional reasoning, teachers should present a variety of tasks where the contexts change and the difficulty of the numerical relationships change.

In an addition to a variety of contexts, students should be exposed to tasks in a variety of forms as well (i.e., symbolic, word problem, and story problem). While many teachers predict that when forms vary students will find the symbolic forms to be the least challenging and the story problem forms the most challenging (due to the amount of words in the problem), imbedded problems in meaningful contexts can help students solve problems. Furthermore, we must remember that symbolic forms require mathematical literacy (e.g., when students are asked to solve $x \cdot 3 + 8 = 30$, many students will first add 3 to 8; whereas, when the problem is imbedded in a meaningful context they will often realize that they first need to subtract 8 from 30). The above examples illustrate problems in various contexts; Table 7.1 shows some examples of problems in various forms where the numerical relationships vary in difficulty.

Table 7.1 – Missing Value problems in various forms*

As a word problem	As a symbolic problem	As a story problem
Some number divided by 2 is equivalent to $\frac{3}{4}$. What is the number?	$\frac{?}{2} = \frac{3}{4}$	A recipe calls for 3 teaspoons of flour for every 4 ounces of water. How many teaspoons of flour should be used with 2 ounces of water?

***We are making a deliberate distinction here between word problems and story problems. We will consider problems embedded within a context as story problems and problems without a context or symbols as word problems as illustrated in the table.

Varying forms, contexts, and numerical relationships can influence the solution strategies that students use. Typically, four different solution strategies are seen when analyzing student work – unit-rate strategy, factor-of-change strategy, fraction strategy, cross-multiplication algorithm (Cramer, Post, 1993). We will illustrate each strategy in Table 7.2 by referring to the story problem in Table 7.1.

Table 7.2 – Four Solution Strategies
(Adapted from Cramer, Post, 1993)

Strategy	Description
Unit-rate strategy	As the name implies, this strategy involves using a unit rate. In the above problem, we are given 3 teaspoons per 4 ounces of water or ¾ of a teaspoon per 1 ounce of water. Students should also be flexible with using the reciprocal of this rate as well depending upon what is being asked for. That is, 4 ounces of water for every 3 teaspoons or 4/3 ounces per 1 teaspoon. In this case, since one is looking for the number of teaspoons for 2 ounces of water, one would take ¾ teaspoon/ounce and multiply it by 2 ounces to obtain 1.5 teaspoons. If the question asked to find ounces of water for a certain number of teaspoons, the other unit rate would be used.
Factor-of-change strategy	A student using a factor-of-change strategy is using a "times as many strategy." In this case, a student would use the following reasoning: Since 4 ounces of water calls for 3 teaspoons of flour and 2 ounces is half of 4 ounces, I need to use half as much flour or 1.5 teaspoons of flour.
Fraction strategy	The fraction strategy is similar to the unit-rate strategy but with the labels dropped on the rates and the idea of equivalence used. In the above story problem, due to the non-integer solution, a fraction strategy may not often be used. If the story problem asked for how many teaspoons of flour are needed for 16 ounces of water, a student using the fraction strategy would reason that ¾ is equivalent to 12/16 by multiplying the numerator and denominator by 4 and conclude that 12 teaspoons are needed. This strategy amounts to using a common denominator approach. A student who has a solid understanding of fractions might reason that one and a half halves is equivalent to three-fourths. Notice that the word problem in Table 7.1 lends itself well to this type of reasoning.
Cross multiplication algorithm	To solve by this strategy a student would set up a proportion (the equivalence of two ratios), find the cross-products, and solve by using division.

While these are typical solution strategies used by students exhibiting proportional reasoning, one also needs to recognize strategies used by students who show misconceptions or transitional type of thinking. The Mr. Tall/Mr. Short problem (Example 7.1) was designed to assess different levels of students' proportional thinking as outlined below in Table 7.3 (Khoury, 2002). We will discuss these levels within the context of the Mr. Tall/Mr. Short problem.

Table 7.3 – Levels of Thinking
(Adapted from Khoury, 2002)

Level	Description
Level 1: Illogical	Illogical computations without an explanation or a general estimate based upon a descriptive observation such as Mr. Tall is 10 paper clips tall since he is really tall and 4 + 6 = 10.
Level 2: Additive	The student focuses on the difference between measurements. For example, Mr. Short is 6 paper clips tall and 4 large buttons tall and 6 – 4 = 2, so Mr. Tall is 6 + 2 or 8 paper clips tall. This may represent one of the most common misconceptions when dealing with proportional relationships and represents students who haven't transitioned from additive reasoning to multiplicative reasoning. Hence, this illustrates the importance of building multiplicative reasoning in the early grades (see Section 5).
Level 3: Transitional	The student uses an additive approach which focuses on the correspondence of the measures. For example, the difference between the number of paper clips and buttons for Mr. Short is 2 and there are 6 paper clips and 4 buttons. So, for each group of 2 buttons there is one more paper clip (i.e., for every 2 buttons there are three paper clips). So, for Mr. Tall, there are (2 + 1) + (2 + 1) + (2 + 1) or 9 paper clips.
Level 4: Ratio	Here the student uses a constant ratio relationship and may involve any of the four strategies discussed above.

Exercise 7.8 Discuss your solution strategies to Exercises 7.3 and 7.4. Go back and solve each of these using a different strategy.

Exercise 7.9 Create some examples of the four different kinds of tasks discussed in this section that can be used to assess proportional reasoning. Make certain to vary the contexts, forms, and numerical relationships.

Limiting students' solution strategies can lead them to use strategies that intuitively do not make sense. Let's examine various strategies for the various forms in Table 7.1.

Solving the story problem in Table 7.1 by using the cross multiplication algorithm leads to the following set-up:

$$\frac{3\text{ teaspoons}}{4\text{ ounces}} = \frac{?\text{ teaspoons}}{2\text{ ounces}}$$

The first step requires multiplying 3 teaspoons by 2 ounces creating an expression without meaning, whereas solving using the unit-rate strategy seems to be a more intuitive approach and more meaningful to the particular situation.

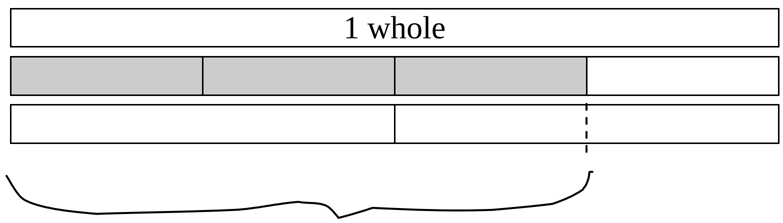
Notice how the symbolic problem in Table 7.1 can be used to model both the word problem and the story problem. However, the interpretation of the symbols changes from the word problem to the story problem. In the story problem, the numerators and the denominators of the fractions have specific meanings (e.g., 3 teaspoons, 4 ounces), whereas, in the word problem one is viewing the fraction 3/4 as a single value and the unknown on the left-hand side as a number to be divided by 2. Using the fraction strategy to solve the word problem seems to be an intuitive approach.

Further caution needs to be taken when determining which quantities are covariant, to determine if the relationship given lends itself to a directly proportional relationship or an indirectly proportional relationship. See Exercise 7.10.

Exercise 7.10 Create a graph and function of the form $y = mx$ by identifying the independent variable, dependent variable, and constant of proportionality (invariant) for the relationship modeled in the story problem in Table 7.1. Repeat the exercise for the word problem in Table 7.1.

Let's take a moment to revisit the symbolic problem in Table 7.1 to see how the relationship defined by that problem can change due to a subtlety in the interpretation of the relationship. In Exercise 7.10, we saw that the word problem defined a directly proportional relationship (i.e., $y = \frac{3}{4}x$, where *y* represents the number to divide by *x* to get

$\frac{3}{4}$ – in other words, one can think of $\frac{3}{4}$ as the invariant, and the dividend (y) and the divisor (x) as covariant). Whereas, if one interprets the relationship shown as the number of one-halves that is equivalent to $\frac{3}{4}$ and defines y to be the number of parts of size x in $\frac{3}{4}$ (see Essential Question 4 from Section 3), the relationship then becomes an indirectly proportional relationship. Investigating this situation (i.e., How many one-halves are equivalent to $\frac{3}{4}$?) by a model seems to be an intuitive approach.



One and one-half halves in three-fourths

We see from this model that as the size of the part increases (the variable x in this situation), the number of parts should decrease and the product of x and y remains constant (i.e., $xy = \frac{3}{4}$). Take a moment to think about how to use the model above to determine how many three-fourths are in one-half (here, one-half is used as the invariant).

Limiting strategies to less intuitive strategies prior to students building a solid conceptual foundation can be detrimental to their understanding of proportionality. Researchers have found that students who have been introduced to the cross-multiplication strategy without a deep conceptual understanding of proportionality will tend to do worse when facing missing-value problems that are non-proportional tasks, as compared to students who haven't been introduced to the cross-multiplication strategy (Cramer, Post, 1993). As an example try Exercise 7.11. Thus, knowing how to perform a procedure does not imply understanding.

Exercise 7.11 Give Essential question 1 (from Section 3) to a group of students who haven't studied the cross multiplication algorithm and to a group of students who have learned the cross multiplication algorithm. Compare the solution strategies and results.

Example 7.5 – Part-to-Part ratios vs. Part-to-whole ratios

When setting up proportions one needs to take caution when given a ratio as to whether or not that ratio represents a part-to-part comparison or a part-to-whole comparison. Read through the examples on the next page from the NECAP Support Materials on ratios.

Example 12.1 - Ratios:

There are 28 students in a fifth grade class. Ten students have blue eyes. Fourteen students have brown eyes. Four students have hazel eyes.

Example of part to whole ratio: The ratio of the students in the class with hazel eyes to the students in the whole class is 4:28 or 1:7. The ratio 1:7 means for every one student with hazel eyes there are 7 students in the class.

Example of a part to part ratio: The ratio of the students in the class with blue eyes to the students in the class with brown eyes is 10:14 or 5:7. The ratio 5:7 means for every 5 students in the class with blue eyes there are 7 students in the class with brown eyes.

Example 12.2 - Demonstrates understanding of ratios (part to whole):

Dana and Jamie ran for Student Council President at Midvale Middle School. The data below represent the voting results for grade 7 and grade 8.

	7th Grade Votes		8th Grade Votes	
	Jamie	Dana	Jamie	Dana
Boys	24	40	25	42
Girls	49	20	19	40

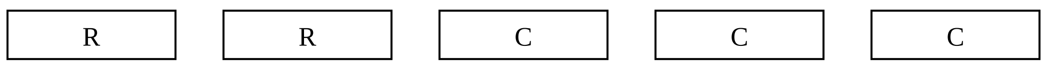
John says that the ratio of the 7th grade boys who voted for Jamie to the seventh grade students who voted for Jamie is about 1:2. Mary disagrees. She says it is about 1:3. Who is correct? Explain your answer.

Answer: Mary is correct. John provided the ratio of boys to girls who voted for Jamie (24 boys:49 girls is about 25:50 or 1:2). Mary provided the ratio of boys who voted for Jamie to all the seventh grade students who voted for Jamie (24 boys:73 seventh grade students is about 25:75 or 1:3).

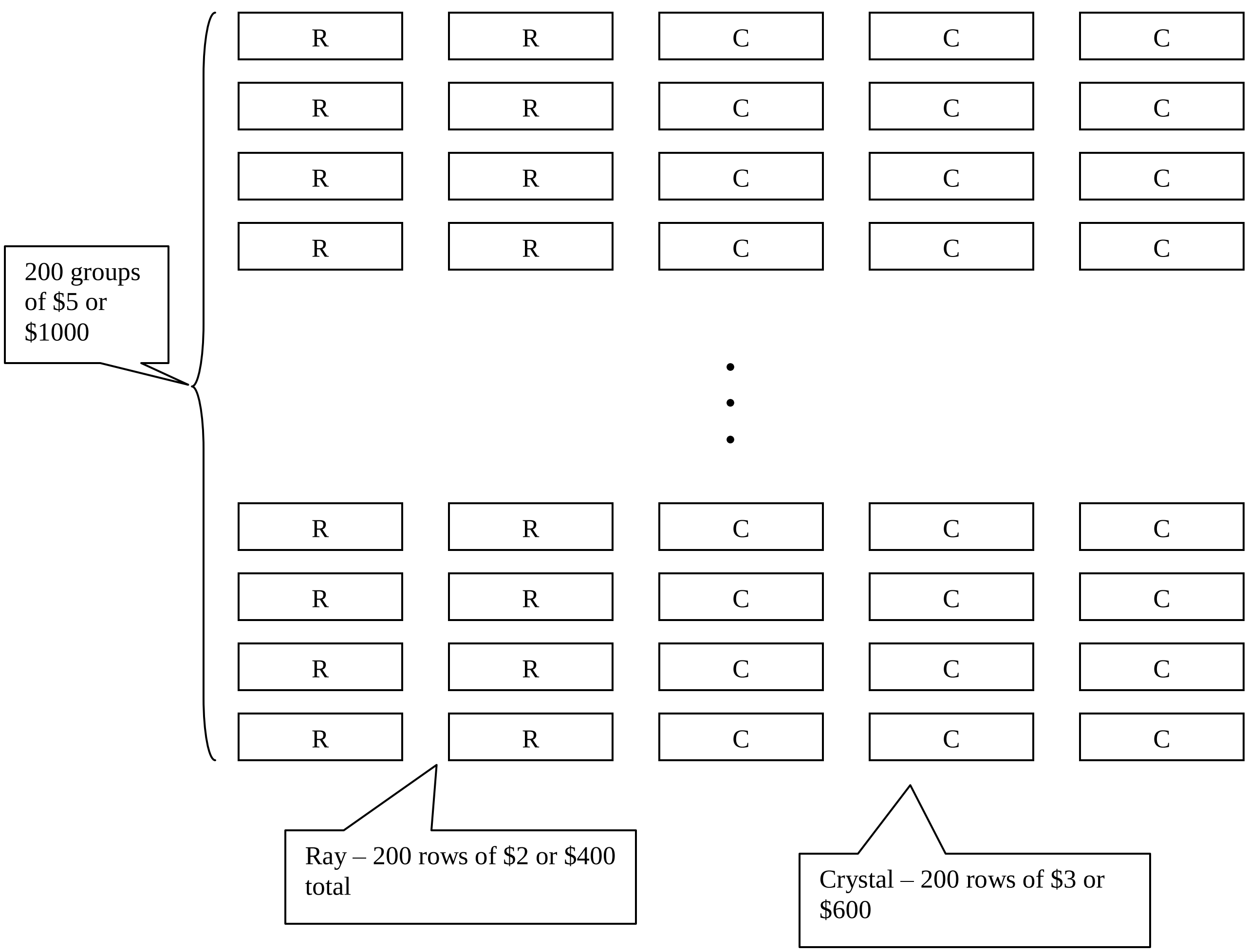
Let’s revisit Essential Question 6 from Section 3.

> **Essential Question 6** Ray and Crystal invested money in a business and will spilt the profit in a ratio of 2:3. If the profit from the business is $1000, how much money will each person receive? Show how to use a model to solve the problem.

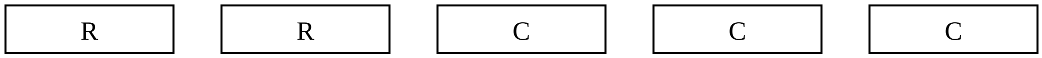
In this question, notice that the ratio 2:3 is a part-to-part ratio. Therefore, some caution needs to be taken when solving this problem. While one can take an algebraic approach to solving this problem, we will illustrate a solution with a model (which may be more intuitive for students building an understanding of proportional reasoning). Notice that the ratio 2:3 means that for every $2 that Ray receives Crystal will receive $3 as illustrated below, where rectangles represent dollar bills and R stands for Ray and C stands for Crystal.



From this model, we can see that for every $5 Ray receives $2 and for every $5 Crystal receives $3. Since there are 1000/5 = 200 groups of $5 in $1000, in total Ray will receive 200 • 2 = $400 and Crystal will receive 200 • 3 = $600. Notice that 400:600 = 2:3 and 400 + 600 = 1000. This is illustrated in the model below.



Alternatively, instructors should help students see that the following model can be used to represent the 2:3 ratio of the profit and the entire profit of $1000.



That is, in this case each  represents $1000/5 or $200. From here, it is easy to see how much Ray receives, $400, and how much Crystal receives, $600.

Exercise 7.12 Solve Essential Question 6 (from Section 3 shown above) by setting up proportions.

Varying the contexts, forms, and numerical relationships and recognizing the various solution strategies, along with beginning instruction with more intuitive strategies, can help students build a deep conceptual foundation of proportional reasoning. Now that we have a better understanding of various assessments tasks and solution strategies, Section 8 will examine student work from released NECAP items, which include items in various contexts.

Section 8: Examining NECAP Released Items and Student Work

This section contains items released from the New England Common Assessment Program, along with some sample student work (for constructed response items) and statistics on the items. Each released item is mapped to a primary Grade-Level Expectation as indicated by the codes next to the items. These Grade-Level Expectations can be located in Section 2. This section will allow you to see how students approach some problems that connect to proportional reasoning. Each NECAP item will be followed by a series of questions that will allow you to view the item within the contexts of this unit of study. Furthermore, the questions following each released item will give you ideas for questions that can be used to probe students' understandings regarding proportional reasoning that connect to the question in the released item. The items and students' responses in this section are reprinted with permission from the New England Common Assessment Program (NECAP).

Exercise 8.1 Read through the following released NECAP item and answer the questions that follow.

NECAP 2005 Released Item – Grade 7, M(N&O)–6–1

1 In an election, for every 4 people who voted for Mr. Smith, 1 person voted for Mr. Jones. Which fraction of the votes did **Mr. Smith** receive?

A. $\frac{1}{4}$

B. $\frac{1}{5}$

C. $\frac{3}{4}$

D. $\frac{4}{5}$

Item Statistics*
(percent of students choosing each option)
A – 35%
B – 8%
C – 25%
D – 31%
*** statistics from 2004 pilot assessment, *n* = 1601**

a) Examine each of the answer options and determine how a student might arrive at each option.

b) Describe how you can use a model to help students see the correct part-to-whole relationship.

c) What additional questions might you ask students who incorrectly answered this question to help them determine their possible misconceptions and lead them to the correct answer?

d) Determine a directly proportional relationship that can be defined from the information given in this item. State which quantities are covariant and determine the invariant.

Exercise 8.2 Read through the following released NECAP item and answer the questions that follow.

10 Look at this table.

Health Club Family Rates

Members in Family	Monthly Cost
1	$ 58
2	$ 88
3	$118
4	$148

Which expression correctly states how the monthly cost is related to the number of members in a family?

A. $30 per person
B. $58 per person
C. $28 plus $30 per person
D. $58 plus $30 per person

NECAP Practice Test – Grade 7, M(F&A)–6–1

Item Statistics*
(percent of students choosing each option)
A – 27%
B – 19%
C – 25%
D – 29%
*** statistics from 2004 pilot assessment, *n* = 1600**

a) Is the relationship given in the table a directly proportional relationship? Explain.

b) Examine each of the answer options and determine how a student might arrive at each option.

c) What percent of the students seemed to treat the relationship in the table as a directly proportional relationship? Explain.

d) What additional questions might you ask students who incorrectly answered this question to help them determine their possible misconceptions and lead them to the correct answer?

Exercise 8.3 Read through the following released NECAP item and student responses and answer the questions that follow.

NECAP 2006 Released Item –
Grade 7, M(G&M)–6–5

⓭ Travis has a photograph that is 4 inches wide and 6 inches tall.

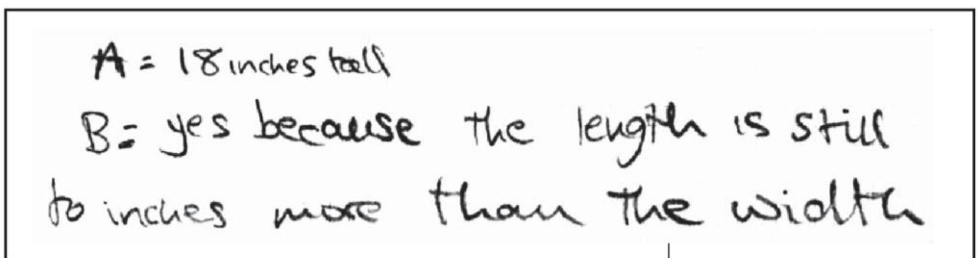
a. Travis enlarges the photograph proportionally so that it is 16 inches wide. How tall is it?

b. Can Travis enlarge the photograph proportionally to 8 inches by 10 inches? Explain your answer.

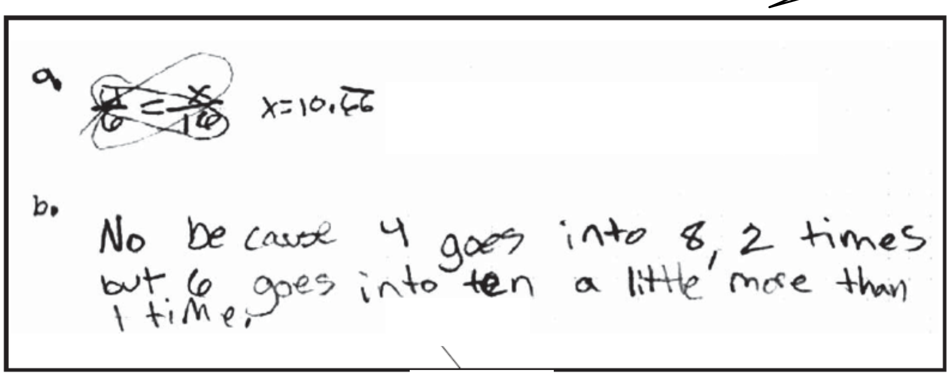
Item Statistics*
(percent of students earning each score point)
0 – 58%
1 – 26%
2 – 16%

***statistics from 2005 NECAP assessment as field test item, *n* = 3915**

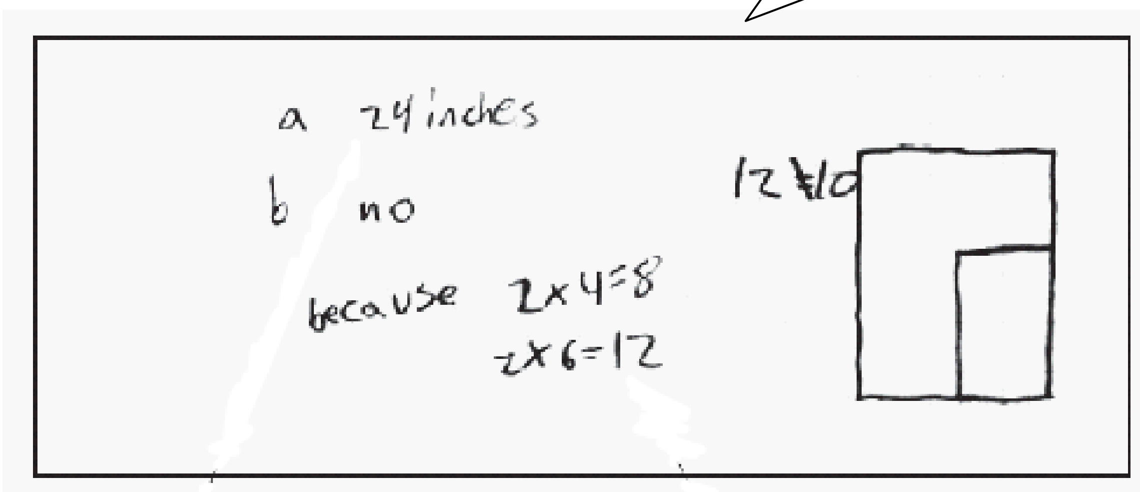
Tom's work



Kris's work



Shawn's work



a) Each part of this item was worth one point (i.e., part a – one point for correct answer, part b – one point for correct explanation). Determine Tom's, Kris's, and Shawn's scores.

b) Using Table 7.3, determine which Level of Thinking (illogical, additive, transitional, ratio) Tom's solution illustrates.

c) Using Table 7.2, which solution strategy (unit-rate, factor-of-change, fraction, cross multiplication) does Kris seem to be using in part a? Which strategy does he seem to be using in part b? Explain.

d) How could Kris's strategy in part b be used to help him successfully solve part a?

e) Using Table 7.2, which solution strategy (unit-rate, factor-of-change, fraction, cross multiplication) does Shawn seem to be using in part b? Explain.

Exercise 8.4 Read through the following released NECAP item and answer the questions that follow.

❻ Dan is filling a swimming pool with water at a constant rate. The table below shows the depth of the water over time.

Time Filling (in hours)	Depth of Water (in feet)
1	$1\frac{1}{2}$
2	2
3	$2\frac{1}{2}$
4	3

How can Dan calculate the depth of the water after 6 hours of filling?

A. Multiply $\frac{1}{2}$ by 6 and add 1.

B. Multiply $1\frac{1}{2}$ by 6.

C. Multiply $2\frac{1}{2}$ by 2.

D. Multiply 1 by 6 and add $\frac{1}{2}$.

NECAP 2006 Released Item – Grade 8, M(F&A)–7–1

Item Statistics*
(percent of students choosing each option)
A – 57%
B – 12%
C – 16%
D – 13%

***statistics from 2005 NECAP assessment as field test item, *n* = 4062**

a) Is the relationship given in the table a directly proportional relationship? Explain.

b) While 16% of students overall chose option C, 25% of the students who scored in the bottom quartile overall on the assessment chose this option. Likely, these students thought that 6 = 3 × 2 so the depth of water should be $2\frac{1}{2} \times 2$. What are some questions you might ask these students to help them see their misconception? When would this type of reasoning result in the correct answer?

Exercise 8.5 Read through the following released NECAP item and Lindsay's response and answer the questions that follow.

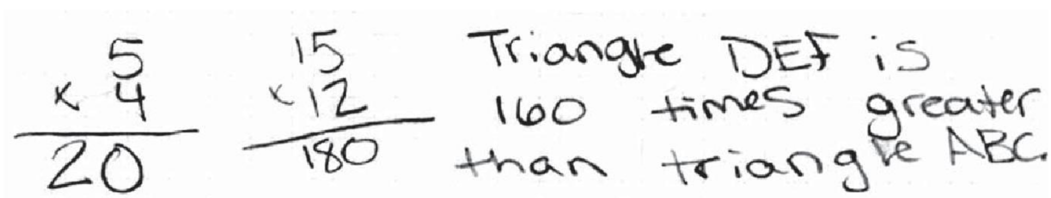
⓮ Triangle *ABC* has a base of 5 inches and a height of 4 inches. Triangle *ABC* is similar to triangle *DEF*. Triangle *DEF* has dimensions that are 3 times as great as those of triangle *ABC*. **How many times as great** is the area of triangle *DEF* compared to the area of triangle *ABC*? Show your work or explain how you know.

NECAP 2006 Released Item –
Grade 8, M(G&M)–7–5

Item Statistics*
(percent of students earning each score point)
0 – 71%
1 – 9%
2 – 19%

***statistics from 2005 NECAP assessment as field test item, *n* = 4024**

Lindsay's response



a) Does Lindsay's response exhibit some understanding of proportional reasoning? Explain.

b) What type of misconceptions does Lindsay's response illustrate?

c) What additional questions might you ask Lindsay to help her get to the correct answer?

d) Show how to determine the answer to the question without directly computing the areas of the triangles.

Exercise 8.6 Read through the following released NECAP item and answer the questions that follow.

7 The equation below shows the profit, p, from selling n cups of lemonade.

$$p = 2n - 10$$

Which of the following best describes the relationship between p and n?

A. As n increases, p decreases.

B. As n increases, p increases.

C. As n increases, p stays the same.

D. As n increases, p sometimes increases and sometimes decreases.

NECAP 2005 Released Item – Grade 8, M(F&A) –7–2

Item Statistics*
(percent of students choosing each option)
A – 18%
B – 53%
C – 12%
D – 15%
***statistics from 2004 pilot assessment, *n* = 1474**

a) Is the relationship between the number of cups sold, n, and the profit p a directly proportional relationship? Explain.

b) What does "10" represent in the equation above? What does "2" represent in the equation above? Describe the meaning of both of these within the context of the problem.

c) The profit from selling 20 cups of lemonade is $p = 2(20) – 10 = 40 – 10 = \$30$. Does this imply that the profit from selling 40 cups of lemonade is double that of selling 20 cups (i.e., $60)? Explain.

d) Use the fact that profit is equal to revenue minus cost, along with your answer to part b, to describe a directly proportional relationship that is implicit in the above situation.

Exercise 8.7 Read through the following released NECAP item and answer the questions that follow.

NECAP 2006 Released Item –
Grade 8, M(N&O)–7–4

2 Katie makes a necklace using the pattern of 2 red beads followed by 3 blue beads. She uses a total of 75 beads for the necklace. How many red beads does Katie use?

A. 25

B. 30

C. 45

D. 50

Item Statistics*
(percent of students choosing each option)
A – 15%
B – 68%
C – 9%
D – 7%
***statistics from 2005 NECAP assessment as field test item, *n* = 4100**

a) How many students from this sample possibly used the three known quantities and one unknown quantity in the problem to set up a proportion and determine the incorrect answer? What additional questions might you ask these students to help them determine if this answer is reasonable?

b) Describe how to solve the problem using a factor-of-change strategy.

c) Describe two quantities, each of which the number of red beads is directly proportional to, and determine the invariant in each case. Express each of these relationships in the form $y = kx$.

Exercise 8.8 There are many other released items that involve proportional reasoning. Examine the NECAP Practice Tests and Released NECAP items from 2005 and 2006 and find other examples of items and student work that connects to proportional reasoning.

Section 9: Summary

Proportional reasoning involves the ability to work fluently with fractions, ratios, rates, and change, and requires a transition from the additive type of reasoning that students study in elementary grades to multiplicative reasoning. Providing students opportunities to think multiplicatively at early grades can help facilitate their understanding of proportions in middle school. Additionally, teachers should strive to provide students opportunities to work with proportionality in a variety of contexts, along with ample opportunities for students to distinguish between proportional and non-proportional situations.

Exercise 9.1 Examine your instructional materials and district curriculum and determine if they provide opportunities for students to apply a variety of solution strategies to proportional relationships (both directly and indirectly proportional) in a variety of contexts and ample opportunities to distinguish between both proportional and non-proportional relationships.

Exercise 9.2 Examine your instructional materials and district curriculum and determine if they provide students ample opportunities to work with proportional relationships that have both integer and non-integer solutions.

Exercise 9.3 Examine your instructional materials and district curriculum and determine if they provide ample opportunities for students to use a variety of representations (e.g., tables, graphs, symbols) when working with proportional relationships and to make connections between them.

Exercise 9.4 Create a curricular sequence/unit of study for the topic of proportionality. The sequence should consider all three principles of how students learn. The following table can serve as a model for your work.

Concept	Description of How Concept is Introduced	Target Depth of Knowledge Levels*	Description of Activities

*You may want to review Appendix A for information on Depth of Knowledge.

This unit of study was meant to introduce you to research around proportionality that can have a direct impact on classroom instruction. It was intended to be used as a supplement to current curricula materials and provide examples that challenge students' understandings of proportional relationships.

If you are looking to build upon the ideas presented in this unit of study and looking to increase your depth of understanding of this vast topic, you are encouraged to explore the references listed in Section 11. Additionally, it is recommended that you research both directly and indirectly proportional relationships, along with joint variation and various higher level applications relating to proportional reasoning such as Newton's Law of Cooling and Population Growth.

Section 10: Answers to Exercises

Section 1: Purpose and Design

Exercise 1.1: Answers will vary.

Section 2: Connecting to the Grade-Level Expectations

Exercise 2.1: Primary GLEs are given on pp. 4–5 of Section 2. Note: Additonal GLEs deal with proportional reasoning – only those that are the main focus of this unit of study are listed.

Exercise 2.2: See the Curriculum Focal Points document from NCTM to find these.

Section 3: Essential Questions

Essential Question 1: 21 laps (sample solution can be found in Section 6)

Essential Question 2: 56 lb (sample solution can be found in Section 6)

Essential Question 3: 2 hr (sample solution can be found in Section 6)

Essential Question 4: $1\frac{1}{2}$ (sample model for a similar problem can be found in Section 7)

Essential Question 5: One would need to know the weights of the individual players to answer this question (sample solution can be found in Section 6)

Essential Question 6: Ray: $400, Crystal: $600 (sample solution can be found in Section 7)

Essential Question 7: No, since the ratios of corresponding sides are not equivalent.

Essential Question 8: Proportional Relationships – 2, 6 (see solution in Section 7 to see which ratios are equivalent), Non-proportional Relationships – 1, 3, 4, 5

(**Note**: See Exercise 4.1 in Section 4 to see which quantities are being identified as covariant for the above solution. For this answer, proportional is taken to mean "directly proportional." Note the discussion in Section 4 and the fact that Essential Questions 3 and 4 are indirectly proportional relationships. See answer to Exercise 4.1.)

Essential Question 9: Answers will vary. Note: constant rate does not mean that a relationship is a directly proportional relationship – it means that there is a linear relationship. For example, the words "constant rate" in Essential Question 1 identify that the relationship is a linear relationship; the words "Kris starting skating first" identify that this linear relationship is not a directly proportional relationship. See discussion in Section 6.

Essential Question 10: Sample answer: 45 laps (See discussion in Section 6.)

Essential Question 11: Linear relationships that pass through the origin (i.e., of the form $y = kx$, for k a non-zero constant) are directly proportional relationships. Linear relationships that do not pass through the origin (i.e., of the form $y = kx + b$, for k and b non-zero constants) are non-proportional linear relationships. See discussion in Section 4.

Essential Question 12: Answers will vary. See discussion in Section 7.

Essential Question 13: The 38 by 34 inch rectangle is more square. The rectangle whose length to width ratio is closest to 1 is the most square.

Essential Question 14: Directly proportional – A and F; Indirectly Proportional – E; Non-proportional – B, C, D, G, H, and I (See Exercise 4.1 and solution in Section 4.)

Essential Question 15: See solution in Section 4 on page 12.

<u>Section 4</u>

Exercise 4.1: Essential Question 8 – 1 and 5 neither, 2 (where *y* represents the weight of *x* bags of mulch and *k* = 7 lb/bag) and 6 (See solution in Section 7 and Exercise 7.12.) directly proportional, 3 (See solution in Section 4.) and 4 indirectly proportional (where *y* represents the number of parts of size *x* in $\frac{1}{2}$); Essential Question 14 – See answer to Essential Question 14.

Exercise 4.2: Answers will vary, but the product of *x* and *y* should be a constant.

<u>Section 5</u>

Exercise 5.1: Answers will vary.

Exercise 5.2: part a) Sample answer: 12 = 2 + 2 + 2 + 2 + 2 + 2, so you will need 5 + 5 + 5 + 5 + 5 + 5 = 30 leaves; part b) Sample answer: 2 + 3 = 5, so you will need 12 + 3 = 15 leaves; part c) Sample answer: If 2 caterpillars need 5 leaves, then 1 caterpillar needs $2\frac{1}{2}$ leaves, so 12 caterpillars need 24 + 6 = 30 leaves; part d) Sample question: You could ask Beth how many leaves 4 caterpillars would need and ask her to draw the situation.

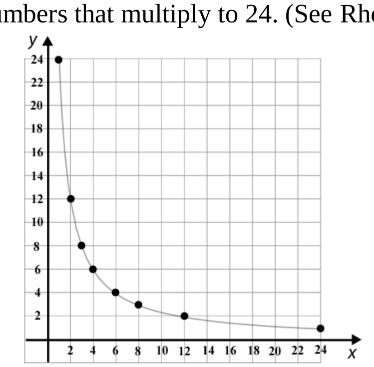
<u>Section 6</u>

Exercise 6.1: See solution in Section 6.

Exercise 6.2: The slope, 1, represents that for each lap Kris completes Rich completes 1 lap.

Exercise 6.3: No; a constant rate of change implies that the relationship is linear. In order to determine if the linear relationship is a proportional relationship, one needs to determine if the relationship can be modeled by a function of the form $y = kx$ for *k* a nonzero constant (or geometrically, directly proportional relationships can be modeled by lines that pass through the origin).

Exercise 6.4: part a) The relationship is not a directly proportional relationship (See discussion preceding this exercise in Section 6); part b) 24 days; part c) 24 person-days; part d) sample solution: Since 2 construction workers is one-third of 6 workers, it should take 2 construction workers 3 times as long to complete the job as 6 construction workers, or 12 days; part e) $y = \frac{24}{x}$; part f) This model is the same for finding two numbers that multiply together to give you 24. The following graph represents this situation. The curve is shown in light gray, while the points represent the whole number solutions that students would find when ask to determine two numbers that multiply to 24. (See Rhode Island Dept. of Ed., 2007.)



Exercise 6.6: Answers will vary.

Exercise 6.7: part a) invariant – 2.5 mi/hr (assuming a constant rate as implied by $d = rt$); time and distance are covariant; 3 hr, 12 min; 4 hr; part b) invariant – 24 mi; rate and time are covariant; 4 mi/hr; 6 mi/hr

Exercise 6.8: part a) Ron's reasoning will not lead to the correct answer since the relationship shown in the table is not a directly proportional relationship; part b) output = $5n + 2$; if the input were 0 then the output would be 2 – to be a directly proportional relationship the output would need to be 0 when the input is 0; the quotient, output/input is not a constant.

Section 7

Exercise 7.1: part a) see answer to Essential Question 13 in Section 3 of the answer section; part b) $y = (38/34)x$ so based on the ratio of the length to width of the first rectangle, if the width changed to 25 inches the length would have to be (38/34)(25) or about 27.9 inches to maintain the squareness of the first rectangle. Since the given length of the second rectangle is 28 inches, it must not be as square as the first rectangle; part c) answers will vary.

Exercise 7.2: part a) Mrs. Keeley's sample will not necessarily contain 5 boys and 6 girls. While there are 5 boys for every 11 students and 6 girls for every 11 students, this does not imply that these ratios hold for each subset of students of Mrs. Keeley's class.; part b) answers will vary – compare the above situation to the lemon mix situation – each sample of the lemonade will contain the same ratio of lemon mix to water.

Exercise 7.3: Measurement problem – 9 paperclips; Speed problem – 11 min, 12 sec

Exercise 7.4: Melanie's pitcher; sample explanation: Melanie's pitcher contains 1 tablespoon for every 8 ounces of water, while Louise's pitcher contains less than 1 tablespoon for every 8 ounces of water since $128 = 8 \times 16$ and she only uses 12 tablespoons.

Exercise 7.5: a

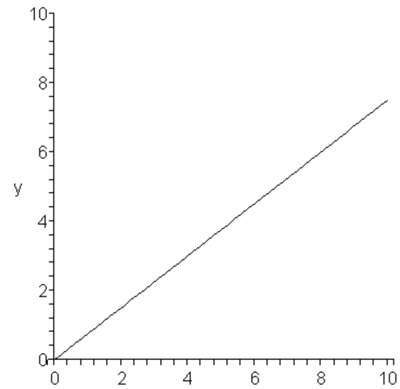
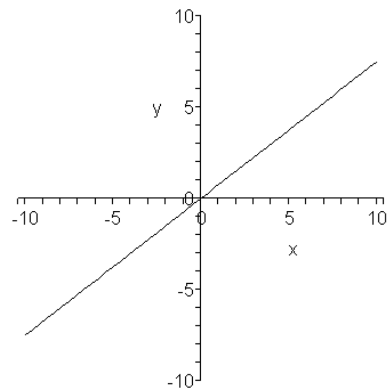
Exercise 7.6: Alisa has a greater number of flowers in a smaller space than Crystal, so Alisa's flowers must be closer together.

Exercise 7.7: part a) Glass A, since adding more mix to Glass A makes the lemon flavor even stronger, while adding more water to Glass B makes the lemon flavor in Glass B weaker; part b) Glass B since it contains less water than Glass A; part c) Glass B, since the same amount of mix was added to both glasses and they contain the same volume with Glass B having the stronger flavor initially; part d) Indeterminate, since we do not know how much stronger the flavor was in Glass B to begin with. As an example, if Glass A initially contains 1 teaspoon of mix and 8 ounces of water and Glass B contains 2 teaspoons of mix and 8 ounces of water, than after the addition of one teaspoon of mix to Glass A and one ounce of water to Glass B, Glass A will contain the stronger mix (1 teaspoon/4 ounces compared to 1 teaspoon/4.5 ounces); whereas, if Glass A initially contains 1 teaspoon of mix and 8 ounces of water and Glass B initially contains 3 teaspoons of mix and 8 ounces of water, then after the additions Glass B will contain the stronger mix (1 teaspoon/3 ounces compared to 1 teaspoon/4 ounces)

Exercise 7.8: Answers will vary.

Exercise 7.9: Answers will vary.

Exercise 7.10:

	Function	**Graph**
Story Problem	$y = \frac{3}{4}x$, where *y* represents the number of teaspoons needed for *x* ounces of water	
Word Problem	$y = \frac{3}{4}x$, where *y* represents the number you divide by *x* to get $\frac{3}{4}$	

Exercise 7:12: If *a* represents the amount of money Ray will receive and *b* represents the amount of money Crystal will receive, the following proportions can be used to solve for *a* and *b*.

$$\frac{2}{5} = \frac{a}{1000}; \qquad \frac{3}{5} = \frac{b}{1000}$$

Section 8

Exercise 8.1: part a) Answers will vary. Sample answers: option A – ratio of votes for Mr. Jones to votes for Mr. Smith, option B – fraction of votes Mr. Jones receives, option C – confuses the part-to-whole relationship, option D – correct answer; part b) Answers will vary. Sample model – see the solution to Essential Question 6 in Section 7 – a similar model can be used here; part c) Questions will vary. Sample questions: What is the ratio of the number of people who voted for Mr. Smith to the number of people who voted for Mr. Jones? What is the ratio of the number of people who voted for Mr. Jones to the number of people who voted for Mr. Smith? If 20 people voted, how many people voted for Mr. Smith and how many voted for Mr. Jones? If 20 people voted, what is the ratio of the number of people who voted for Mr. Smith to the total number of people who voted?; part d) Answers will vary. Sample answer: The number of votes that Mr. Smith receives, *y*, is directly proportional to the number of people voting, *x*, and can be related by $y = \frac{4}{5}x$.

Exercise 8.2: part a) No, the quotient of the monthly cost and the number of members in the family is not constant or alternatively, there is a set fee of $28 a month to use the health club regardless of the number of members in the family; part b) Answers will vary; part c) Those students who answered A or B likely treated the relationship as a proportional relationship or about 46% of the students in the sample; part d) Questions will vary. Sample questions: What is the increase in the cost from one member to two members? From two members to three members? From three members to four members? Is there a monthly cost to be a member of the health club? Using the pattern in the table, what would be the monthly costs if there were zero members in the family? Explain what this means in terms of the context of the problem.

Exercise 8.3: part a) 0 points, 1 point; 2 points; part b) Level 2: Additive; part c) Kris's grouping of the means and the extremes of his proportion indicate that he is probably using the cross multiplication algorithm in part a. Kris seems to be using the factor-of-change strategy in part b; part d) Kris might reason that 4 goes into 16 four times so the correct answer is 24 since 6 goes into 24 four times or Kris might notice that 6 goes into 10.66 a little more than one time while 4 goes into 16 four times so his answer isn't reasonable; part e) Shawn seems to be using the factor-of-change strategy since he shows that 8 is 2 times four and therefore to be proportional the height would need to be 2 times 6 or 12 inches.

Exercise 8.4: part a) No, by the relationship given in the table, the initial depth of the water was 1 foot; part b) Questions will vary. Sample questions: Based on the relationship in the table, what was the depth of water in the swimming pool before Dan began filling it? What might be the depth of the water in the pool after 5 hours? By how much does the depth of the water increase each hour?; this type of reasoning would result in a correct answer if the relationship in the table were a directly proportional relationship.

Exercise 8.5: part a) Lindsay seems to have some understanding of a factor-of-change strategy as evident by her ability to find the dimensions of triangle *DEF*; part b) Lindsay reverts to additive reasoning when answering how many times as great the area of triangle *DEF* is compared to the area of triangle *ABC*. Additionally, while one can not be sure, it appears that Lindsay may believe that the area of a triangle is equal to the length of its base times the length of its height – there is no evidence that Lindsay understands why one can leave the ½ out of the problem; part c) Questions will vary. Sample questions: How did you determine the dimensions of triangle *DEF*? How can you use this reasoning to help you determine how many times greater the area of triangle *DEF* is compared to the area of triangle *ABC*? If triangle *DEF* had a base of 25 inches and a height of 20 inches, how many times as great would the dimensions of triangle *DEF* be compared to the dimensions of triangle *ABC*? part d) Sample answer: If the length of the base of triangle *ABC* is *b* and the height of triangle *ABC* is *h*, then the area, *A*, of triangle *DEF* is $A = \frac{1}{2}(3b)(3h)$ which is 9 times the area of triangle *ABC*.

Exercise 8.6: part a) No, since the relationship is not of the form $p = kn$ for some constant k or alternatively, the graph of the relationship does not pass through the origin; part b) 10 represents some initial starting costs – that is, if no cups of lemonade are sold one will lose $10; 2 represents the slope or the change in profit for each cup of lemonade sold – that is, the profit increases by $2 for each cup of lemonade sold; part c) No, one could only double the profit from 20 cups to find the profit for 40 cups if the relationship was a directly proportional relationship; part d) The revenue is directly proportional to the number of cups sold with the invariant being 2 – that is, revenue = 2 × number of cups sold.

Exercise 8.7: part a) likely 287 students (those who answered option D); Questions will vary. Sample questions: Does Katie use more red beads or blue beads in her design? If 50 out of the 75 beads Katie used were red beads, how many blue beads did Katie use? Based on your answer, what would the ratio of the number of red beads used to the number of blue beads be?; part b) Answers will vary. Sample answer: the number of total beads used is $2\frac{1}{2}$ times the number of red beads used, so if Katie uses 75 total beads she must have used 30 red beads since $30 \times 2\frac{1}{2}$ is 75; part c) The number of red beads, y, is directly proportional to the number of blue beads, x, and the relationship can be expressed as $y = \frac{2}{3}x$; the number of red beads, y, is also directly proportional to the total number of beads, x, and can be expressed as $y = \frac{2}{5}x$.

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Appendix A – Implied Cognitive Demand and Depth of Knowledge

A fundamental criterion used to develop the NECAP GLEs and GSEs is that the expectations should explicitly indicate cognitive demand (how content interacts with process) and that there should be a mix of cognitive demand levels at all grades. That is, one should not assume that students at lower grades do less cognitively demanding work. The cognitive demand or depth of knowledge required by an expectation or an assessment item is related to the number and strength of connections of concepts and procedures that a student needs to make to produce a response, including the level of reasoning required along with self-monitoring. Furthermore, there are additional factors that influence cognitive demand including contextual requirements, language, the number and variety of representations, requirements for generalizations to new situations, and the opportunity to learn.

It is important to note that depth of knowledge is not synonymous with difficulty. As an example, solving a multi-step linear equation with variables on both sides may be a difficult task for middle school students; however, the task can be solved by applying a standard procedure making the task of low complexity.

The NECAP states believe that expectations and assessment should be aligned in terms of their cognitive complexity. That is, the cognitive complexities of the assessment items should match that of the standards (what students are expected to know and be able to do). To ensure this alignment, the NECAP states have adopted Norman L. Webb's (senior researcher with the Wisconsin Center for Educational Research) Depth of Knowledge classification system. Norman Webb's system is based on four levels of classification. The full descriptions of each level are given on pages 4 and 5. The levels can be summarized as follows.

Level 1	Recall
Level 2	Skill/Concept
Level 3	Strategic Thinking
Level 4	Extended Thinking

The NECAP states, together with a committee of educators, analyzed the GLEs and GSEs for their implied cognitive demand. That is, all aspects of each expectation were analyzed and the implied cognitive demand levels were recorded. One of the charges of the NECAP test item review committees is to ensure that assessment items align not only with the expectations but also with their implied cognitive demands. The range of cognitive demands for each GLE and GSE is summarized in Table 1 on page 2. It should be noted that the highest level listed for each GLE and GSE should be thought of as a "ceiling" not a "target". That is, the goal is to write items which cover the range of the levels indicated and not just the highest level. If one assesses only at the "target" level, all GLEs with a level 3 (for example) as their highest cognitive demand would only be assessed at level 3. This would potentially have two negative impacts on the assessment: 1) The assessment as a whole would be too difficult; and 2) important information about student learning along the achievement continuum would be lost. To the extent possible, GLEs and GSEs should be assessed at the "ceiling" and at least one level below the "ceiling" in order to provide additional diagnostic information to educators. Furthermore, Table 2 shows an example of an expectation and how the different aspects of the expectation interact with Table 1.

Table 1

	Depth of Knowledge Levels for NECAP Assessment						
	2	**3**	**4**	**5**	**6**	**7**	**10**
M(N&O)–X–1	1, 2	1, 2	1, 2	1, 2	1, 2	1, 2	
M(N&O)–X–2	1	2	2	2	2	2	1, 2, 3
M(N&O)–X–3	1, 2	2	2	2,3	2,3		
M(N&O)–X–4		1, 2, 3	1, 2, 3	1, 2, 3	1, 2, 3	1, 2, 3	1, 2, 3
M(N&O)–X–5	1, 2						
M(G&M)–X–1	1, 2, 3	1, 2	1, 2	1, 2	1, 2		
M(G&M)X–2						1, 2	1, 2, 3
M(G&M)–X–3			1, 2	1, 2	1, 2		
M(G&M)–X–4				1, 2		1, 2	2, 3
M(G&M)–X–5			1, 2		1, 2	1, 2, 3	1, 2, 3
M(G&M)–X–6	1, 2	1, 2	1, 2	1, 2	1, 2, 3	1, 2, 3	1, 2, 3
M(G&M)–X–7	This GLE will NOT be directly assessed but embedded in problems in other content strands.						1, 2
M(G&M)–X–8							
M(G&M)–X–9							2, 3
M(F&A)–X–1	2	2	2	2	2, 3	2, 3	2, 3
M(F&A)–X–2					1, 2	1, 2, 3	1, 2, 3
M(F&A)–X–3			1	1	1, 2	1, 2	1, 2
M(F&A)–X–4	1	1, 2	1, 2	1, 2	1, 2	1, 2	
M(DSP)–X–1	1, 2, 3	1, 2, 3	1, 2, 3	1, 2, 3	1, 2, 3	1, 2, 3	2, 3
M(DSP)–X–2	2, 3	2, 3	2, 3	2, 3	2, 3	2, 3	2, 3
M(DSP)–X–3		1, 2		1, 2		2, 3	1, 2, 3
M(DSP)–X–4	2		2, 3		2, 3		1, 2, 3
M(DSP)–X–5		1, 2	1, 2	1, 2	1, 2, 3	1, 2, 3	1, 2, 3

Black cells indicate GLEs or GSEs that are not assessed on NECAP at the given level.

Sample Mathematics GLE* for End of Grade 6	Potential DoK Levels	DoK Ceiling	Aspects of GLE at different levels**
M(F&A)–6–1 **Identifies and extends to specific cases a variety of patterns** (linear and nonlinear) represented in models, tables, sequences, graphs, or in problem situations; or writes a rule in words or symbols for finding specific cases of a linear relationship; or writes a rule in words or[sc] symbols for finding specific cases of a nonlinear relationship; and writes an expression or[sc] equation using words or[sc] symbols to express the **generalization** of a linear relationship (e.g., twice the term number plus 1 or[sc] 2*n* + 1).	**2, 3**	**3**	Level 2 **Extends a pattern to a specific case** **Level 3** Generalizes a pattern

Table 2

*GLE NOTES: Underlining in the GLE indicates that this concept or skill is "new" to grade 6 for assessment purposes. The superscript "sc" indicates that students have a choice in how they complete the task (e.g., students can use words **or** symbols to express the rule).

**Recall, one must also consider other factors when making decisions on Depth of Knowledge levels such as contextual requirements, language, the number and variety of representations, requirements for generalizations to new situations, and the opportunity to learn.

Depth of Knowledge Descriptors for Mathematics
Norman L. Webb
March 28, 2002

Mathematics Depth of Knowledge Levels

Level 1 (Recall) includes the recall of information such as a fact, definition, term, or a simple procedure, as well as performing a simple algorithm or applying a formula. That is, in mathematics a one-step, well-defined, and straight algorithmic procedure should be included at this lowest level. Other key words that signify a Level 1 include “identify,” “recall,” “recognize,” “use,” and “measure.” Verbs such as “describe” and “explain” could be classified at different levels depending on what is to be described and explained.

Level 2 (Skill/Concept) includes the engagement of some mental processing beyond a habitual response. A Level 2 assessment item requires students to make some decisions as to how to approach the problem or activity, whereas Level 1 requires students to demonstrate a rote response, perform a well-known algorithm, follow a set procedure (like a recipe), or perform a clearly defined series of steps. Keywords that generally distinguish a Level 2 item include “classify,” “organize,” ”estimate,” “make observations,” “collect and display data,” and “compare data.” These actions imply more than one step. For example, to compare data requires first identifying characteristics of the objects or phenomenon and then grouping or ordering the objects. Some action verbs, such as “explain,” “describe,” or “interpret” could be classified at different levels depending on the object of the action. For example, if an item required students to explain how light affects mass by indicating there is a relationship between light and heat, this is considered a Level 2. Interpreting information from a simple graph, requiring reading information from the graph, also is a Level 2. Interpreting information from a complex graph that requires some decisions on what features of the graph need to be considered and how information from the graph can be aggregated is a Level 3. Caution is warranted in interpreting Level 2 as only skills because some reviewers will interpret skills very narrowly, as primarily numerical skills, and such interpretation excludes from this level other skills such as visualization skills and probability skills, which may be more complex simply because they are less common. Other Level 2 activities include explaining the purpose and use of experimental procedures; carrying out experimental procedures; making observations and collecting data; classifying, organizing, and comparing data; and organizing and displaying data in tables, graphs, and charts.

Mathematics Depth of Knowledge Levels continued

Level 3 (Strategic Thinking) requires reasoning, planning, using evidence, and a higher level of thinking than the previous two levels. In most instances, requiring students to explain their thinking is a Level 3. Activities that require students to make conjectures are also at this level. The cognitive demands at Level 3 are complex and abstract. The complexity does not result from the fact that there are multiple answers, a possibility for both Levels 1 and 2, but because the task requires more demanding reasoning. An activity, however, that has more than one possible answer and requires students to justify the response they give would most likely be a Level 3. Other Level 3 activities include drawing conclusions from observations; citing evidence and developing a logical argument for concepts; explaining phenomena in terms of concepts; and using concepts to solve problems.

Level 4 (Extended Thinking) requires complex reasoning, planning, developing, and thinking most likely over an extended period of time. The extended time period is not a distinguishing factor if the required work is only repetitive and does not require applying significant conceptual understanding and higher-order thinking. For example, if a student has to take the water temperature from a river each day for a month and then construct a graph, this would be classified as a Level 2. However, if the student is to conduct a river study that requires taking into consideration a number of variables, this would be a Level 4. At Level 4, the cognitive demands of the task should be high and the work should be very complex. Students should be required to make several connections—relate ideas *within* the content area or *among* content areas—and have to select one approach among many alternatives on how the situation should be solved, in order to be at this highest level. Level 4 activities include designing and conducting experiments; making connections between a finding and related concepts and phenomena; combining and synthesizing ideas into new concepts; and critiquing experimental designs.

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