



Understanding Equality

A Research Based Unit of Study for Elementary School Teachers

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Section 1: Purpose and Design

Equality is a keystone topic in elementary school mathematics and receives a high degree of emphasis in the New England Common Assessment Program (NECAP) Grade-Level Expectations (GLEs). Additionally, how students begin to view equality in elementary school has a direct affect on how they view equality throughout their school years and has a profound impact on students' initial understanding of algebra. *Understanding Equality* is one unit of study in a series of units designed around the keystone topics in elementary school mathematics.

The purpose of this unit of study is to support quality instruction by increasing teachers' mathematical content knowledge primarily through examining research findings related to students' understandings of equality, along with examining student work. Furthermore, the topics in these units of study are mathematical areas in which researchers have identified many student misconceptions. When adapting the ideas in these units of study to the classroom, teachers should purposefully design their lesson plans to incorporate three important principles of learning as identified by the National Research Council in *How Students Learn – Mathematics in the Classroom* and described in Table 1.1 on the following page (Donovan & Bransford, 2005). It should also be noted that this unit is not meant to supplant current curricular materials but rather to be used in conjunction with them. Furthermore, the three principles described in Table 1.1 are modeled throughout this unit of study. The Essential Questions in Section 3 seek to engage preconceptions, the various sections throughout the unit connect procedural knowledge to conceptual knowledge, and there are checkpoints along the way to monitor and reflect on your progress. Additionally, the summary section (Section 9) contains exercises allowing you to reflect upon your instructional programs and curricular materials.

Each unit of study is broken into sections that build upon one another. Furthermore, each unit of study begins by examining the Grade-Level Expectations that are pertinent to the particular unit of study. Following the identification of the expectations related to the unit of study, you will answer some essential questions. It is recommended that you answer these essential questions individually prior to reading subsequent sections. The essential questions will help frame the mathematical ideas of the unit. Exercises appear throughout the remaining sections. These exercises are imbedded within the section rather than at the end of the section and are intended to be solved and discussed as you are working through the section. Section 8 is devoted to examining NECAP released items and the student work that is available for these items. This section is subsequent to the sections that discuss research findings. So, once you reach this section, you will be able to make connections between the research and the NECAP items and identify typical student misconceptions.

This unit of study does not attempt to cover all areas of equality, but rather is an introduction into the research behind the aspects of this topic that elementary school students study. Ideas for content extensions and research extensions can be found by exploring the references in Section 11.

Additionally, it is worth noting that students need opportunities to work collaboratively, share ideas, and present ideas. Their understandings should be challenged and students should be allowed to build and construct knowledge for themselves. Students should actively engage in mathematics. Teachers should carefully guide this work, and should be cautious about just telling students the ‘answers.’ Teachers should ask students to explain how they know and allow them to share multiple ways to solve problems. Teachers need to continually probe students’ understandings, especially by allowing students to explore new ideas on their own. Teachers need to resist modeling a few dozen low-level problems (i.e., not cognitively challenging) for students and then asking them to solve far too many homework problems all of which can be matched to one of the modeled problems.

Table 1.1 – Principles of Learning

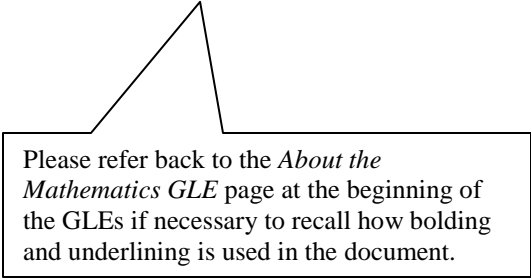
Principle	Description
<i>Principle 1 – Engaging Preconceptions</i>	Students come to the mathematics classroom with ideas about the structures of mathematics and informal understandings. If their preconceptions are not engaged and if there is no bridge between informal and formal understanding, students may have difficulty learning new ideas and may continue to revert to their preconceived notions.
<i>Principle 2 – Connecting Procedural/factual knowledge and Conceptual Understanding</i>	Procedural knowledge and conceptual understanding must be balanced. When one places too much emphasis on procedural fluency the result is a lack of understanding in how the procedures work. Whereas, when one places too much emphasis on conceptual knowledge, often students lack the ability to perform the procedures in an efficient way. Teachers must help students build and connect ideas and organize knowledge into networks. It is important to discuss various solution methods and why they work and make connections among them.
<i>Principle 3 – Self Monitoring</i>	Students need to be afforded the opportunity to think about their own learning and assess their own mathematical progress. Eventually, such assessment opportunities will be internalized and students will begin to self-monitor their own progress.

Section 2: Connecting to the Grade-Level Expectations

Equality permeates the GLEs and represents a big idea in elementary school mathematics. At this level, students begin to develop initial understandings of the equal sign along with properties of equality, the properties associated with whole numbers and positive fractional numbers, and the operations on them. One of the goals of this unit of study is to help you become more familiar with the Grade-Level Expectations in elementary grades dealing with equality, including the research behind them. In order to achieve this goal, you will spend some time looking through the Grade-Level Expectations (see Exercise 2.1) to find the standards related to equality.

In Section 8 you will spend time analyzing released NECAP items and student work associated with these items. Please complete Exercise 2.1 before reading on.

Exercise 2.1 Locate the GLEs in elementary grades that deal with equality and properties of equality. Make certain that you are considering both state and local GLEs.



Please refer back to the *About the Mathematics GLE* page at the beginning of the GLEs if necessary to recall how bolding and underlining is used in the document.

Exercise 2.2 Locate the Curriculum Focal Points and Connections at elementary grades that deal with equality and properties of numbers and the operations on them.

While many GLEs are connected to equality, this unit of study will primarily deal with content contained within the following GLEs (New Hampshire Department of Education & Rhode Island Department of Education, *NECAP and Local Mathematics Grade-Level Expectations for grades K-8*, 2005).

Grade 2

M:N&O:2:8 Applies properties of numbers (odd and even) and **field properties** (commutative for addition, identity for addition, and associative for addition) **to solve problems and to simplify computations involving whole numbers.**

M:F&A:2:4 Demonstrates conceptual understanding of equality by finding the value that will make an open sentence true (e.g., $2 + \square = 7$). (limited to one operation and limited to use addition or subtraction)

Grade 3

M:N&O:3:8 Applies properties of numbers (odd, even, and multiplicative property of zero for single-digit whole numbers [$6 \times 0 = 0$]) and **field properties** (commutative for addition, associative for addition, identity for multiplication, and commutative for multiplication for single-digit whole numbers [e.g., $3 \times 4 = 4 \times 3$]) **to solve problems and to simplify computations involving whole numbers.**

M:F&A:3:4 Demonstrates conceptual understanding of equality by showing equivalence between two expressions using models or different representations of the expressions; or by finding the value that will make an open sentence true (e.g., $2 + \square = 7$). (limited to one operation and limited to use addition, subtraction, or multiplication)

Grade 4

M:N&O:4:8 Applies properties of numbers (odd, even, multiplicative property of zero, and remainders) and **field properties** (commutative, associative, and identity) **to solve problems and to simplify computations.**

M:F&A:4:4 Demonstrates conceptual understanding of equality by showing equivalence between two expressions using models or different representations of the expressions, by simplifying numerical expressions where left to right computations may be modified only by the use of parentheses [e.g., $14 - (2 \times 5)$] (expressions consistent with the parameters of M:F&A:4-3), and by solving one-step linear equations of the form $ax = c$, $x \pm b = c$, where a , b , and c are whole numbers with $a \neq 0$.

Grade 5

M:N&O:5:8 Applies properties of numbers (odd, even, and divisibility) and **field properties** (commutative, associative, identity, and distributive) **to solve problems and to simplify computations.**

M:F&A:5:4 Demonstrates conceptual understanding of equality by showing equivalence between two expressions using models or different representations of the expressions (expressions consistent with the parameters of M:F&A:5-3), by solving one-step linear equations of the form $ax = c$, $x \pm b = c$, or $x/a = c$, where a , b , and c are whole numbers with $a \neq 0$; or by determining which values of a replacement set make the equation (multi-step of the form $ax \pm b = c$ where a , b , and c are whole numbers with $a \neq 0$) a true statement (e.g., $2x + 3 = 11$, $\{x: x = 2, 3, 4, 5\}$).

Section 3: Essential Questions

Essential questions help you to begin to think about the mathematics that will be the focus of this unit of study. You are encouraged to think deeply about these questions and to work them independently before discussing your thoughts with colleagues and before reading subsequent sections. It is also recommended that you keep a journal that contains your work on these problems and the problems throughout this unit of study. You are encouraged to use PEN in your journal so that you can not easily erase your work. Even though this is contrary to what many mathematics teachers ask students to do, using pen will allow you to go back and reflect on your initial thoughts, analyze any errors that you have made, and see how your learning has developed. (This is a suggestion that Tim Kurtz, NH State Assessment Director, has passed on to me that I try to share whenever possible. As a teacher, if you require your students to use pen, you will be able to easily identify what students were thinking when working problems and any errors made by students. This will facilitate your efforts in addressing students' preconceptions and misconceptions and will allow students to monitor their progress – See Table 1.1.) Some of these questions are intentionally vague in some areas. The reasons for the intentional vagueness will be apparent when one works through the remainder of the sections. Many of these questions will be discussed throughout various sections of this unit of study; therefore, please refrain from looking at the answers to the essential questions (Section 10) until working through the entire unit of study. Additional questions will be posed throughout the various sections.

Essential Question 1 Is the following statement true or false? Explain.

$$3 + 7 = 10 + 5 = 15 + 2 = 17$$

Essential Question 2 Determine the number that belongs in the box and the number that belongs in the circle in the equality statement below. Explain how you found the values for the box and the circle.

$$22 + 18 = \square + 12 = \circ$$

Essential Question 3 Determine two typical incorrect answers that students give when asked to find the number that belongs in the box below. Explain your choices.

$$8 + 5 = \square + 2$$

Essential Question 4 Describe, in your own words, what the symbol “=” means.

Essential Question 5 Give a few examples of where you have seen students misuse or misinterpret the symbol “=”. Explain.

Essential Question 6 Describe as many situations as you can where the “=” sign could be used in different ways.

Essential Question 7 Do “equal” and “equivalent” mean the same thing? Explain.

Essential Question 8 Describe, in your own words, the Commutative Property of Addition. Support your description with examples, and draw a model to illustrate the property.

Essential Question 9 Describe, in your own words, the Commutative Property of Multiplication. Support your description with examples, and draw a model to illustrate the property.

Essential Question 10 Describe, in your own words, the Associative Property of Addition. Support your description with examples, and draw a model to illustrate the property.

Essential Question 11 Describe, in your own words, the Associative Property of Multiplication. Support your description with examples, and draw a model to illustrate the property.

Essential Question 12 Describe, in your own words, the Distributive Properties. Support your description with examples, and draw a model to illustrate the properties.

Essential Question 13 Explain what is meant by relational thinking as it pertains to students solving problems involving equality like Essential Question 3. Describe some various strategies to engage students in relational thinking.

Essential Question 14 Is there any difference between an unknown and a variable? Explain.

Essential Question 15 Find all values for the symbols in the following statement that make the equality true.

$$\square + \square = 12$$

Find all values for the symbols in the following statement that make the equality true.

$$\square + \Delta = 12$$

Describe how the above questions are similar or different.

Essential Question 16 How do you think students would respond to the task below? What questions might they have about the value of \square on each side of the equation? (Adapted from Carpenter, Franke, & Levi, 2003)

Find all values for \square that make the following statement true.

$$\square + 3 = \square + 8$$

Essential Question 17 Use properties to justify the following statement (assume $b \neq 0$).

$$a \times b \div b = a$$

Describe how a student would justify the above statement. Give some concrete examples.

As you can see from our essential questions, some careful consideration needs to be given to what the symbol “=” means, along with how students approach equality. We probably don’t spend enough time challenging students’ conceptions of equality. As a result, students harbor many misconceptions about equality and subsequently struggle with this concept in algebra courses. Carefully building students’ understandings of equality through arithmetic and the operations that students use will help students develop the foundations that they need to be successful in K–12 mathematics and beyond. The next section will review how the term “equality” and how the phrase “Demonstrates conceptual understanding of equality by showing equivalence” is used throughout the GLEs by examining the NECAP support materials. Section 5 will approach equality and the properties of equality from a much more theoretical perspective than that presented in the NECAP support materials. Section 5 is intended to give you a much deeper understanding of the mathematics behind a topic that we often treat as trivial. Subsequent sections to Section 5 will examine students’ understandings of equality and properties of equality from a teaching perspective.

Section 4: Equality and the NECAP Support Materials

This section explores what is meant in the GLEs by “equality” and “Demonstrates conceptual understanding of equality by showing equivalence” (see the targeted GLEs listed in Section 2) by examining the NECAP support materials. This section will help exemplify some of the ways in which students work with equality in elementary grades. The next section will examine equality and properties of equality from a more theoretical standpoint. Subsequent sections will expand on the ideas in this section by examining students’ understandings of equality and properties of equality, along with examining these topics from a teaching perspective.

Exercise 4.1 Read through the excerpts from the NECAP support materials shown below, and work through the examples. Revisit your responses to Essential Questions 4–6. (Note: The number below is that which is used in the NECAP support materials.)

F&A – 30 Equality: Equality refers to the condition in which two expressions that have the same value are joined by an equal sign. These two expressions are called equivalent expressions.

F&A – 31 Demonstrates conceptual understanding of equality by showing equivalence: To demonstrate conceptual understanding of equality by showing equivalence means to illustrate that two expressions joined by an equal sign have the same value. Each example below shows a way in which students will be asked to demonstrate understanding of equality by showing equivalence.

Example 31.1: Find the value that will make an open sentence true (Grade 1 and up)

$$2 + \square = 7$$

$$\square = 5$$

Example 31.2: Show that the expressions connected by the equal sign have the same value using models or other representations of the expressions (Grade 3 and up)

$$4 + 5 = 3 + 6$$

$$\blacksquare\blacksquare\blacksquare\blacksquare + \blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare = \blacksquare\blacksquare\blacksquare + \blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare$$

$$\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare = \blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare$$

$$9 = 9$$

Example 31.3: Use models to show equivalence (Grade 3 and up)
Each of these shapes represents a value.



What is the value for each shape in the number sentences below? Each shape stands for the same value no matter where it is placed.

$$\square + \square + \square = 12$$

$$\square + \bullet + \blacktriangle = 12$$

$$\blacktriangle + \blacktriangle + \blacktriangle + \blacktriangle = 12$$

Answer: ▲ = 3; □ = 4; ● = 5

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Section 5: Equality, Equivalence Relations, and Properties of Equality

The word equal can be described as referring to being the same or identical in some way. That is, in some way two quantities are equivalent to each other. Throughout this section we will examine various properties of equality and the properties that apply to various sets that students work with (e.g., the set of natural numbers) and the operations on them. This section is intentionally more theoretical in nature than the remaining sections. The purpose of this section is to help you develop a deeper understanding of what we mean by equivalent, along with properties of equality. This section introduces you to two important mathematics topics – Rings and Fields. At first, the section may seem dense. However, the mathematics introduced in this section will help you begin to develop a deeper understanding of an area of mathematics that we often feel is trivial for students – equality. While we will not develop all the ideas in this section as carefully as we could, it will give you a better sense of the various properties that students work with. Do not give up on the remaining sections of this unit of study due to this section. In fact, the unit is designed so that it may be completed without working through all of the details of this section. However, it is recommended that you attempt all of the exercises, read and re-read the section, and revisit the section as necessary. It will take some time for these concepts to become clear if you haven't studied them formally before. The topics in this section are **not** meant to be introduced to students in the way that they are here. Subsequent sections will address these topics from a teaching perspective and examine students' understandings of equality along with how students build an understanding of the properties that are addressed in this section. Furthermore, we will revisit some of the exercises in this section from a teaching perspective. We begin this section by revisiting Essential Question 6.

Exercise 5.1 Take some time to revisit your response to Essential Question 6 from Section 3 shown below. Share your ideas with your colleagues before reading on.

Essential Question 6 Describe as many situations as you can where the “=” sign could be used in different ways.

Example 5.1 Table 5.1 describes three different ways the equal sign is used throughout K–12 mathematics.

Table 5.1 – Three Different Ways the Equal Sign is Used in K-12 Mathematics*

Description	Examples
<i>Conditional Equality</i>	A conditional equality situation is a situation such as $5 + \square = 12$. In this situation, one is looking for a value for \square that makes the equation true. The equation is only true in a very limited sense – that is, only when the value of \square is 7. Other values make the equality false.
<i>Identically Equivalent Expressions</i>	Identically equivalent expressions are expressions that result in the same value no matter what we choose for the value of the variable. For example $x + x = 2x$. This statement is true no matter what the value of x . As another example, $5 + 3 = 8$ is an example of identically equivalent expressions.
<i>Definitional Equality</i>	Definitional equality occurs when two quantities are equal by definition. For example, defining $y = 2x$.

*While in practice, it is common to use the symbol $=$ in all three of the cases described in Table 5.1, there is mathematical notation to distinguish the second and third cases from the first. The symbol \equiv is commonly used to mean identical to (e.g., $x + x \equiv 2x$); whereas, the symbol $:=$ is commonly used to mean equal by definition (e.g., $y := 2x$).

One of the most important things to understand about equality is that the equality relation “ $=$ ” forms an equivalence relation. An equivalence relation on a set A is a relation R that satisfies three properties: the reflexive property, the symmetric property, and the transitive property as shown in Table 5.2.

Table 5.2 – Equivalence Relation Properties

Property	Description
<i>Reflexive Property</i>	For all x in some set A , $x R x$.
<i>Symmetric Property</i>	For all x and y in some set A , if $x R y$ then $y R x$.
<i>Transitive Property</i>	For all x , y , and z in some set A , if $x R y$ and $y R z$, then $x R z$.

At first this notation may be a little confusing, but it can easily be clarified by looking at an example and defining what the relation R is.

Example 5.2 Let the relation R be “knows” and A be the set of all people in the world. Obviously the reflexive property holds. That is, every person in the world knows himself or herself. (Given person x knows given person or $x R x$ where x represents the given person and R is the “knows” relation.) Similarly, if person 1 knows person 2 ($x R y$) then it is certainly the case that person 2 knows person 1 ($y R x$). However, the transitivity property fails, so the relation “knows” is not an equivalence relation. The verification of this is left as the next exercise.

Exercise 5.2 Explain why the transitivity property does not hold for the “knows” relation.

While it may be familiar to you that the relation “=” is an equivalence relation, students often have a hard time with the symmetric property and transitive property.

Example 5.3 – Equality as an Equivalence Relation

The equality relation “=” satisfies three important properties.

Reflexive Property: For all x in some set A , $x = x$.

Symmetric Property: For all x and y in some set A , if $x = y$ then $y = x$.

Transitive Property: For all x , y , and z in some set A , if $x = y$ and $y = z$, then $x = z$.

As an example of the difficulty that students may have with the symmetric property when using composition and decomposition of numbers, consider the relation “=” and the set A to represent the set of real numbers. Research shows that students often have difficulty with the following question due to their viewing the equal sign in an operational way rather than in a relational sense. (See Section 6 for a discussion on this item.)

True or False: $8 = 5 + 3$

While many students are comfortable with the fact that $5 + 3 = 8$, they often have a difficult time seeing that $8 = 5 + 3$ since they believe that the “answer” can not be on the left-hand side. Purposeful instruction that probes students’ understandings around these three properties will help build a solid foundation for important mathematics to come. Also note that the equal sign in this case is being used in an identically equivalent way. (See Table 5.1.)

Exercise 5.3 Let R be the relation \leq and A be the set of whole numbers, $\{0, 1, 2, 3, 4, \dots\}$. Is this relation an equivalence relation? That is, does the relation \leq satisfy all three properties in Table 5.2? Explain.

In addition to the reflexive, symmetric, and transitive properties it is important to understand the properties that apply to the sets that students work with (e.g., whole numbers, natural numbers, real numbers) and the operations on them (+ and \times).

Understanding these properties can give us insight into how students use operations to solve problems and which operations and properties are appropriate.

A non-empty set together with two operations is called a *Ring* if the properties in Table 5.3 are satisfied. (Note: We will write these two operations as $+$ and \times , and refer to them as addition and multiplication, but these operations are not always the traditional addition and multiplication as illustrated in Example 5.6.)

Table 5.3 – Ring Properties*

Property	Description (for $a, b,$ and c elements of a set S)
<i>Closure for Addition</i>	If a and b are elements of a set S , then $a + b$ is an element of the set S .
<i>Associative for Addition</i>	$a + (b + c) = (a + b) + c$
<i>Commutative for Addition</i>	$a + b = b + a$
<i>Additive Identity or Zero Element</i>	There exists an element 0 in S such that $a + 0 = a$ and $0 + a = a$ for all a in S .
<i>Additive Inverses</i>	For each element a in S , there exists an inverse element in S , denoted by $-a$, such that $a + (-a) = 0$. In other words, for each a in S the equation $a + x = 0$ has a solution that is an element of S .
<i>Closure for Multiplication</i>	If a and b are elements of a set S , then $a \times b$ is an element of S .
<i>Associative for Multiplication</i>	$a \times (b \times c) = (a \times b) \times c$
<i>Distributive Properties</i>	$a \times (b + c) = a \times b + a \times c$ and $(a + b) \times c = a \times c + b \times c$

*Some definitions require a ring to have a multiplicative identity as well.

Example 5.4 The set of integers, $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$, with the usual addition and multiplication forms a ring.

Exercise 5.4 Take some time to discuss why the integers form a ring.

Example 5.5 The set of odd integers does not form a ring. This can be proved by showing that the Closure for Addition Property does not hold. That is, the sum of two odd integers is not another odd integer. Why? Take a moment to verify this. How might students use manipulatives to show that the sum of two odd integers is an even integer?

Exercise 5.5 Explain why the set of odd integers does not have an additive identity element.

Exercise 5.6 Is the set of even integers a ring? Explain.

Example 5.6 Define a new addition, denoted by the symbol Δ (take a moment to look at M:N&O:8:8), and a new multiplication, denoted by the symbol \blacktriangle on the set of integers by

$$a \Delta b = a + b - 1 \text{ and } a \blacktriangle b = a + b - ab$$

where the addition, subtraction, and multiplication on the right-hand side of equal signs is ordinary addition, subtraction, and multiplication. To verify that the Commutative Property of Addition holds, remembering that the symbol Δ denotes addition, one would need to show that

$$a \Delta b = b \Delta a.$$

And, since by definition $a \Delta b = a + b - 1$ and the addition on the right-hand side of the equal sign is ordinary addition on the integers which is known to be commutative, we can switch the order of a and b on the right hand side to obtain

$$a \Delta b = a + b - 1 = b + a - 1.$$

Finally, since by definition $b \Delta a = b + a - 1$ we have

$$a \Delta b = a + b - 1 = b + a - 1 = b \Delta a.$$

You may want to try some integers for a and b to get a feel for these new operations.

This verifies that the Commutative Property of Addition holds.

Example 5.7 Take addition and multiplication defined on the set of integers as described in Example 5.6. To find the additive identity or zero element, one needs to find an integer (an element of the set of integers) that we denote by 0 such that

$$a \Delta 0 = a \text{ and } 0 \Delta a = a \text{ for all integers } a.$$

Some caution needs to be taken here. Just because we denoted the additive identity element by 0, this does not mean that this element is the integer 0. In fact, it may be helpful to call the element that we are looking for m for missing. With this in mind, we are looking for an integer m such that

$$a \Delta m = a \text{ and } m \Delta a = a \text{ for all integers } a.$$

Then, by definition we have

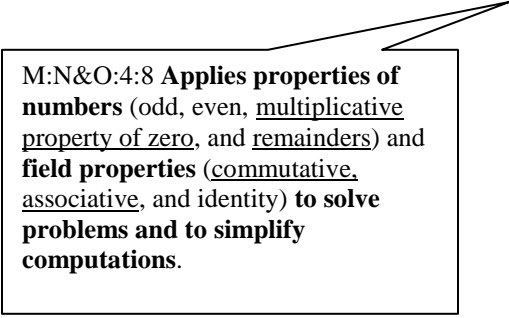
$$a \Delta m = a + m - 1 \text{ and } m \Delta a = m + a - 1.$$

Thus, we can see that the missing element m that we are looking for to get $a \Delta m = a$ and $m \Delta a = a$ for all integers a is the integer 1. (Replace m with 1 in the above equations to verify.) Therefore, the additive identity element for the set of integers with addition and multiplication defined as in Example 5.6 is 1.

Exercise 5.7 In Example 5.7 we showed that the additive identity element for the set of integers with addition and multiplication defined as shown in Example 5.6 is 1. Knowing that this is the additive identity or “zero” element, find the additive inverse for any element a in this set.

Exercise 5.8 Verify that the set of integers with addition and multiplication defined as shown in Example 5.6 is a ring.

You may notice that there are some properties that you are familiar with that are not shown in Table 5.3 (e.g., Commutative for Multiplication or a Multiplicative Identity). Some of these properties are also referenced in the GLEs as field properties.



M:N&O:4:8 **Applies properties of numbers** (odd, even, multiplicative property of zero, and remainders) and **field properties** (commutative, associative, and identity) **to solve problems and to simplify computations.**

Every field is a ring but not every ring is a field. A field is essentially a ring with some additional properties that hold as illustrated in Table 5.4. Notice that the first eight properties are the ring properties.

Table 5.4 – Field Properties

Property	Description (for $a, b,$ and c elements of a set S)
<i>Closure for Addition</i>	If a and b are elements of a set S , then $a + b$ is an element of the set S .
<i>Associative for Addition</i>	$a + (b + c) = (a + b) + c$
<i>Commutative for Addition</i>	$a + b = b + a$
<i>Additive Identity or Zero Element</i>	There exists an element 0 in S such that $a + 0 = a$ and $0 + a = a$ for all a in S .
<i>Additive Inverses</i>	For each element a in S , there exists an inverse element in S , denoted by $-a$, such that $a + (-a) = 0$. In other words, for each a in S the equation $a + x = 0$ has a solution that is an element of S .
<i>Closure for Multiplication</i>	If a and b are elements of a set S , then $a \times b$ is an element of S .
<i>Associative for Multiplication</i>	$a \times (b \times c) = (a \times b) \times c$
<i>Distributive Properties</i>	$a \times (b + c) = a \times b + a \times c$ and $(a + b) \times c = a \times c + b \times c$
<i>Commutative for Multiplication</i>	$a \times b = b \times a$
<i>Multiplicative Identity</i>	There exists an element 1 in S such that $a \times 1 = a$ and $1 \times a = a$ for all a in S .
<i>Multiplicative Inverses</i>	For each nonzero element a in S , there exists an inverse element in S , denoted by a^{-1} , such that $a \times a^{-1} = 1$ and $a^{-1} \times a = 1$. In other words, for each nonzero a in S the equations $a \times x = 1$ and $x \times a = 1$ have a solution that is an element of S .

Additionally, every field satisfies a property known as No Zero Divisors as shown in Table 5.5.

Table 5.5 – No Zero Divisors*

<i>No Zero Divisors</i>	There are no proper divisors of 0. That is, if $a \times b = 0$, then $a = 0$ or $b = 0$.
-------------------------	---

*Algebra I textbooks often refer to this property as the Zero-Product Rule.

While the No Zero Divisor Property may not be as familiar as the other properties, we are familiar with the fact that the set of integers satisfies this property. That is, if two integers have a product of 0, then one of the integers is 0. Additionally, it is used fairly often by students in algebra class when working with polynomials and especially when trying to solve a quadratic equation such as $x^2 + 5x + 6 = 0$. Often, students will solve this equation by factoring $x^2 + 5x + 6$ into $(x + 3)(x + 2)$ and solving the resulting equation $(x + 3)(x + 2) = 0$ by setting each factor equal to 0 (i.e., concluding that $x + 3 = 0$ or $x + 2$

= 0). While you may not have worked with factoring or multiplying polynomials for some time, there are some nice connections to arithmetic.

Before reading on, please take a moment to think about how multiplication of two numbers (e.g., two two-digit whole numbers) connects to the area of a rectangle, and try to show how the partial products form the area of the rectangle.

If it has been a while since you have worked with multiplying polynomials or factoring polynomials, one can verify that $(x + 3)(x + 2) = x^2 + 5x + 6$ by applying the Distributive Property. (Thinking of $a = x + 3$, $b = x$, and $c = 2$ – one should note that a , b , and c are now polynomials.) Applying the distributive property with a , b , and c defined as shown above we have:

$$(x + 3)(x + 2) = (x + 3)(x) + (x + 3)(2).$$

We now repeat the Distributive Property twice on the right-hand side of the above equation. We first use the Distributive Property on the first piece, $(x + 3)(x)$ thinking of $a = x$, $b = 3$, and $c = x$. Then, we use the Distributive Property on the second piece, $(x + 3)(2)$ thinking of $a = x$, $b = 3$, and $c = 2$. Now we have,

$$(x + 3)(x + 2) = (x + 3)(x) + (x + 3)(2) = x(x) + 3x + x(2) + 3(2) = x^2 + 5x + 6.$$

Note, that we used the Commutative Property to replace $x(2)$ with $2(x)$ before combining $3x$ and $2x$ to make $5x$. Also, please note that we are skipping over many details of polynomials including the fact that we are using x two different ways – as an indeterminate in the above multiplication and as an element of the real numbers in the above equation. This discussion and distinction isn't necessary to this unit of study. You are encouraged to see Hungerford, 1990 to explore this topic further.

Notice that the analogous procedure is done when multiplying two-digit by two-digit numbers.

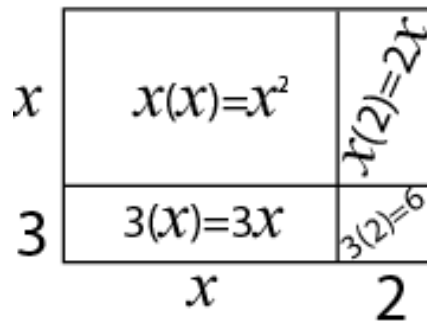
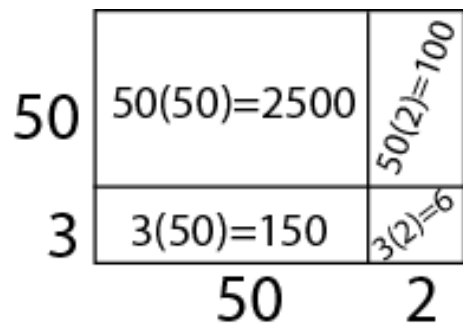
$$\begin{array}{r} 53 \\ \times 52 \\ \hline 2500 \text{ (partial product } 50 \times 50) \\ 150 \text{ (partial product } 3 \times 50) \\ 100 \text{ (partial product } 50 \times 2) \\ \underline{6 \text{ (partial product } 3 \times 2)} \\ 2756 \end{array}$$

Writing out in an analogous way to $(x + 3)(x + 2)$ using the Distributive Property we have:

$$53 \times 52 = (50 + 3)(50 + 2) = (50 + 3)(50) + (50 + 3)(2) = (50)(50) + 3(50) + 50(2) + 3(2).$$

[Note: You may recall the term F.O.I.L. being used to help you to remember how to multiply binomials. The F.O.I.L. procedure lends little understanding to the process of multiplying polynomials. To build conceptual understanding, teachers should avoid using this term and teach the distributive property.]

We can describe the Distributive Property geometrically for both of the examples in the box on the previous page by thinking of multiplication of two numbers as representing the area contained within a rectangle. (Note: the following drawings are not to scale.)



Algebra can be described as generalized arithmetic. When teaching arithmetic, we should seek to present ideas in a way that will foster algebraic thinking. If you wish to practice multiplying polynomials and factoring polynomials using manipulatives, please explore the Algebra Tile virtual manipulative from the National Library of Virtual Manipulatives that can be located at http://nlvm.usu.edu/en/nav/category_g_3_t_2.html.

Example 5.8 – The Set of Real Numbers

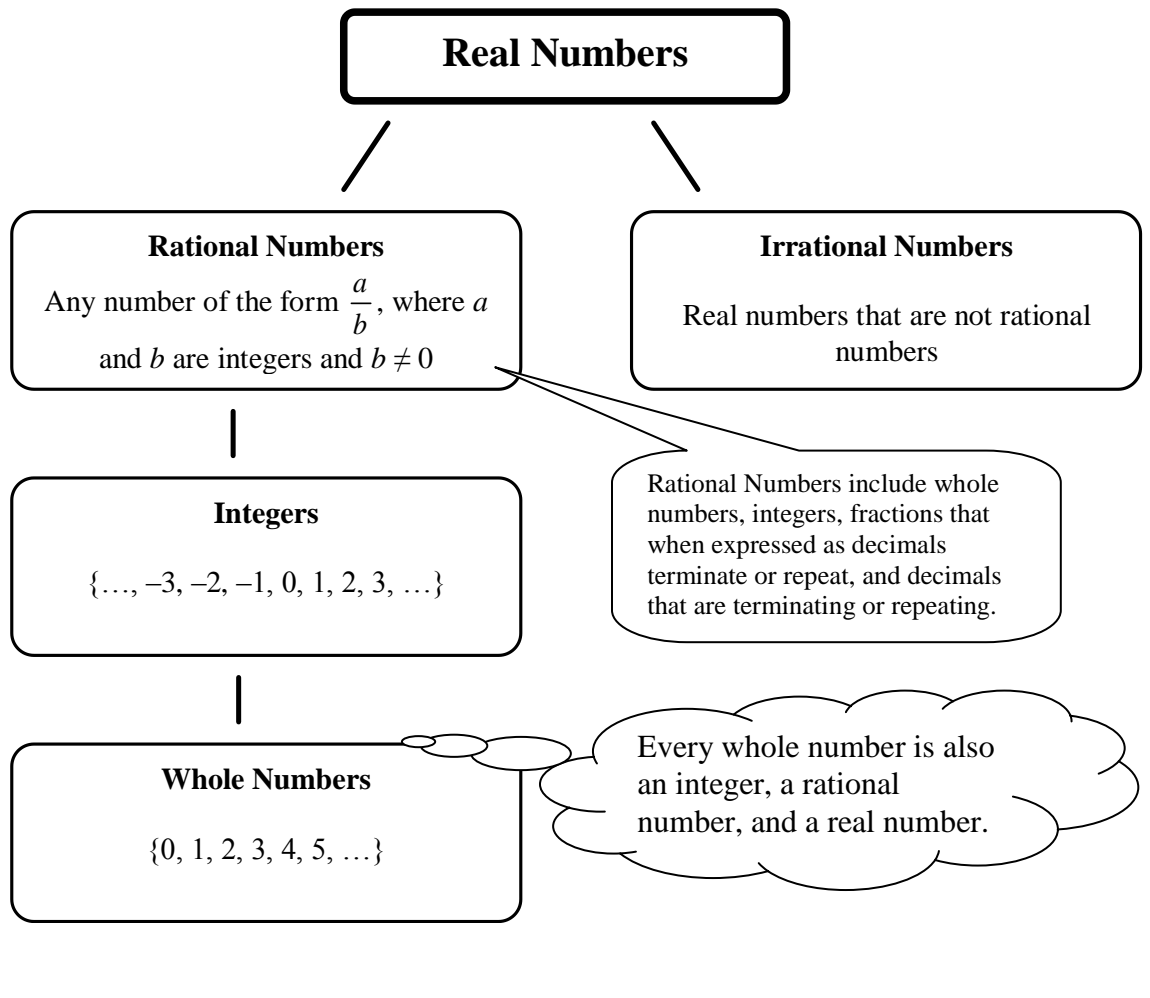
The set of real numbers with the usual addition and multiplication is a field. See the next page for a description of the real numbers as illustrated in the NECAP support materials.

N&O – 11 Real Numbers: Real numbers include rational and irrational numbers.

The following diagram shows how the subsets of the real number system (that have been defined in this section) are related to each other.

The diagram shows, for instance, that every whole number is also an integer, a rational number, and a real number; however, not every integer is a whole number (e.g., -3 is an integer, but not a whole number); not every rational number is an integer (e.g., $\frac{2}{5}$ is a rational number, but not an integer); no irrational number is a rational number; every integer is a rational number (e.g., $2 = \frac{2}{1}$), and so on.

Relationships Between Some of the Subsets of the Real Number System



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Example 5.9 – The Set of Integers

The set of integers with the usual addition and multiplication is not a field. Given any nonzero integer there does not exist a multiplicative inverse that is an integer. For example, there is no integer for which you multiply the integer 2 by to get the identity, 1.

Exercise 5.9 Verify that the set of rational numbers (see the definition on the previous page) with the usual addition and multiplication is a field.

Exercise 5.10 Consider the set of integers with addition and multiplication defined as shown in Example 5.6. Find the multiplicative identity element. Be careful to clearly indicate any properties that you are using that hold for the operations on the right-hand side. (That is, properties that hold for ordinary addition and multiplication of integers.) Don't forget to make certain that your multiplicative identity element is an integer. Additionally, see Example 5.7 to recall what the additive identity element is and verify that the additive identity does not equal the multiplicative identity.

When one refers to Properties of Equality, it is somewhat vague which properties one is speaking of. Furthermore, since not all of the properties in Table 5.4 hold for all sets with an addition and multiplication operation defined that students will work with throughout their K-12 mathematics careers, some caution needs to be taken regarding when specific properties hold (e.g., students in high school study matrices and the Commutative Property of Multiplication does not hold). However, in addition to the properties that have been discussed above, there are other common properties that students work with. These additional properties of equality play a fundamental role in algebra and are shown in Table 5.6.

Table 5.6 – Some Common Properties of Equality*

Property	Description
<i>Substitution Property of Equality</i>	If $a = b$, then a can be replaced by b wherever a occurs and vice versa.
<i>Addition Property of Equality</i>	If $a = b$, then $a + c = b + c$.
<i>Subtraction Property of Equality</i>	If $a = b$, then $a - c = b - c$.
<i>Multiplication Property of Equality</i>	If $a = b$, then $ac = bc$.
<i>Division Property of Equality</i>	If $a = b$ and $c \neq 0$, then $a/c = b/c$.

* The Substitution Property of Equality holds by the definition of equality. The Addition, Subtraction, Multiplication, and Division Properties follow directly from the Substitution Property of Equality.

We are beginning to see the natural development of the hierarchy of the real numbers (See page 19.). That is, if one is only working in the realm of natural numbers and trying to solve equations of the form $a + \uparrow = b$, the solution $b - a$ only makes sense if $b > a$ (e.g., when working with only natural numbers and trying to solve the equation $3 + \uparrow = 2$ one finds that there is no natural number solution). Hence, extending the natural numbers to the integers by introducing the symbols $0, -1, -2, -3, \dots$, along with the definition that $b - a = -(a - b)$ for $b < a$ allows us to solve this subtraction issue (i.e., one adds meaning to subtracting a larger number from a smaller number). For example, in the above equation, $3 + \uparrow = 2$, $b - a = 2 - 3 = -(3 - 2) = -1$. In a similar way, extending the integers to the rational numbers allows one to solve equations of the form $ax = b$. Equations of this form will only have solutions that are integers when a is a factor of b (e.g., $2x = 8$ has an integer solution since 2 is a factor of 8).

Example 5.10 – The Distributive Property and $-1 \times -1 = 1$

The distributive property, $a(b + c) = ab + ac$, can help us understand perplexing issues such as why $-1 \times -1 = 1$ and not -1 . If we want the distributive property to hold then it must be the case that negative one times negative one is positive one; for, if it were the case that negative one times negative one was negative one, then according to the distributive property, with $a = -1$, $b = 1$, and $c = -1$, we have

$$-1(1 - 1) = -1 - 1 = -2.$$

Whereas, if one performs the subtraction in the parentheses first one obtains $-1(0) = 0$. (Adapted from Courant & Robbins, 1996)

Example 5.11 – Defining Subtraction

Up until Table 5.6 you might have noticed that we hadn't specifically talked about subtraction. In fact, the ring properties in Table 5.3 are defined for operations that we call addition and multiplication. However, we know from the Additive Inverse Property that for each a in S the equation $a + x = 0$ has a solution that is an element of S . We also indicated in Table 5.3 that we denote this solution by $-a$. You may have questioned whether or not this solution is unique or not. That is, is there more than one value for x such that $a + x = 0$ for a particular a in S ? Let's suppose that there are two values for x , call these values r and s , that will make $a + x = 0$ true for a particular value of a . Then, we would have:

$$a + r = 0 \text{ and } a + s = 0.$$

Using the fact that $s = 0 + s$ we would then get the following:

$$s = 0 + s = (a + r) + s = (r + a) + s = r + (a + s) = r + 0 = r.$$

Thus, there must be an unique solution. Hence we can call this unique solution $-a$. This allows us to define subtraction in any ring to mean:

$$a - b = a + (-b) \quad \textbf{Subtraction in any Ring}$$

Example 5.12 – Some familiar properties

Now that we know how to define subtraction in a ring, we are able to prove many familiar properties for elements a and b of a ring R such as those described in Table 5.7 below.

Table 5.7 – Some familiar Properties

Property	Proof
$a \cdot 0 = 0 \text{ and } 0 \cdot a = 0$	<p>Since 0 is the additive identity element we know that $0 + 0 = 0$. Using the substitution property of equality we have,</p> $a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0.$ <p>Again, by the additive identity property we know that $a \cdot 0 + 0 = a \cdot 0$. Using the Substitution Property of Equality to replace $a \cdot 0$ on the left-hand side of the first statement with $a \cdot 0 + 0$ we have,</p> $a \cdot 0 + 0 = a \cdot 0 + a \cdot 0.$ <p>Subtracting $a \cdot 0$ from both sides (See Exercise 5.12.) gives us $0 = a \cdot 0$ or $a \cdot 0 = 0$. The proof for $0 \cdot a = 0$ is similar.</p>
$a \cdot (-b) = -(a \cdot b) = -a \cdot b$	<p>We showed in Example 5.11 that the additive inverse of each element of a ring is unique. Thus, $-(a \cdot b)$ is the unique solution to the equation $a \cdot b + x = 0$. We will show that $a \cdot (-b)$ is also a solution to this equation and thus $a \cdot (-b) = -(a \cdot b)$. Using the Distributive Property (and really the Symmetric Property of equality on the first Distributive Property given in Table 5.4), the Additive Inverse Property, and the first property in this table we have,</p> $a \cdot b + a \cdot (-b) = a \cdot (b + (-b)) = a \cdot 0 = 0.$ <p>The proof of the other part is similar.</p>
$-(-a) = a$	<p>Similar to the previous proof we use that fact that $-(-a)$ is the unique solution to the equation $-a + x = 0$. However, a is also a solution so $-(-a) = a$.</p>
$(-a)(-b) = ab$	<p>This can be shown by using the second and third properties in this table.</p>
<p>For a ring with an Identity, $-1 \cdot a = -a$</p>	<p>By the second property in this table and the Multiplicative Identity Property we have,</p> $-1 \cdot a = -(1 \cdot a) = -a.$

The properties in Table 5.7 can be explored using algebra tiles. Before reading on, take some time to think about how you might explain the properties in Table 5.7 to students. Please explore the first sixteen slides of the *Let's Do Algebra Tiles* PowerPoint at <http://www.delmar.edu/aims/> under the presentations link, and connect the use of the algebra tiles to the proofs presented in Table 5.7 and your ideas on presenting these concepts to students.

Exercise 5.11 Knowing that subtraction can be defined in terms of additive inverses, use the field properties in Table 5.4 to justify the following statement. Also, take some time to describe how a student would justify the statement.

$$a + b - b = a$$

Exercise 5.12 In the proof of the first property of Table 5.7, we used the converse of the Addition Property of Equality from Table 5.6. That is,

$$\text{If } a + c = b + c, \text{ then } a = b.$$

Show that this is true for elements a , b , and c in any ring R , by using the Addition Property of Equality to add $-c$ to both sides of $a + c = b + c$ followed by the use of Ring Properties.

A similar argument to the one presented in Example 5.11 can be used to define division in terms of multiplicative inverses when working with a field.

$$a \div b = a \cdot b^{-1} \quad \text{Division in any Field}$$

While subtraction can be defined in terms of additive inverses and division in terms of multiplicative inverses, that doesn't mean that we should just do away with these operations. They both provide natural ways to model many problem situations and computationally are useful when dealing with large numbers (Carpenter, Franke, & Levi, 2003).

Exercise 5.13 Let's revisit Essential Question 17.

Essential Question 17 Use properties to justify the following statement.
(Assume $b \neq 0$.)

$$a \times b \div b = a$$

Describe how a student would justify the above statement. Give some concrete examples.

Knowing that division can be defined in terms of multiplicative inverses, revisit your solution to this essential question and check to see if you used the field properties in Table 5.4 to justify the statement. Also, take some time to revisit your description of how a student would justify the statement.

Exercise 5.14 Show that in a field multiplicative inverses are unique.

Exercise 5.15 Does the Distributive Property hold over subtraction? That is, does $a \times (b - c) = (a \times b) - (a \times c)$? Explain. You may want to use some properties from Table 5.7.

Exercise 5.16 Does the Commutative Property hold over subtraction? Over division? Does the Associative property hold over subtraction? Over division? Explain. Support your conclusions with some examples.

Exercise 5.17 Determine if the following statement is true for all real numbers a , b , and c . Explain.

$$a \times (b \times c) = a \times b + a \times c$$

Section 6 will examine students' understandings of equality and properties of equality. Before reading on, take some time to revisit Essential Questions 8–12 now that you have spent some time working with these properties. Solutions to these exercises will be discussed in Section 6.

Section 6: Students' Understandings of Equality

This section will revisit the topics of Section 5 from a teaching perspective. We will look at students' preconceptions and misconceptions regarding equality and spend time analyzing how students work with the various properties in Table 5.4. We will also spend time discussing tasks that will challenge students' understandings of equality.

Even though teachers typically introduce the equal sign in a relational way as indicating a balance or equivalence between two expressions, research shows that young children tend to form their own meaning for $=$. Often, this meaning is to interpret the $=$ sign in an operational manner, much as they would interpret the $+$ or $-$ sign. That is, the equal sign is often viewed by children as signifying that it is time to write the answer (Ginsburg & Baron, 1993).

Despite teachers best intentions to introduce the $=$ sign in a relational sense, it isn't surprising that students interpret the symbol as the time to write the result to some problem. Without making intentional decisions to stress equivalence, without having deliberate conversations with students regarding their understandings of the $=$ sign, and without intentionally challenging students' understandings of $=$, students are likely to interpret the symbol in the way that they are most familiar using it – in problems where the symbol does signify the end of the process such as mental mathematics exercises (e.g., students are asked to calculate 3 plus 4 in their heads) or exercises such as finding the sum of two or more addends (e.g., $3 + 4 = \square$). Additionally, the use of the “ $=$ ” sign on calculators to signify the end of a process may contribute to students' misunderstandings of the symbol. However, researchers have shown that students at very young ages, with limited exposure to any type of number sentences, strongly believe that the equal sign signifies the end of a calculation. The good news is that researchers have found that young children can learn the appropriate use of the equal sign, and their understandings do not depend on their facility with computation (Carpenter, Franke, & Levi, 2003).

Designing questions to help elicit students' understandings of equality can be an informative exercise. Furthermore, research indicates that students clarify their notations of equality when their existing conceptions are challenged. Table 6.1 describes four benchmarks of equality that researchers have identified which can serve as a framework for designing questions that intentionally challenge students' conceptions of equality (Carpenter et al., 2003 as described in Henderson & Christensen, 2006). Before looking at Table 6.1, please take the time to work through the following ten exercises (adapted from Henderson & Christensen, 2006).

Exercise 6.1 Work through the following ten questions.

In questions 1–3, fill in the box with the correct number. Explain how you know that the number you put in the box is correct.

- 1) $8 + 5 = \square + 2$
- 2) $47 + 21 = 45 + \square$
- 3) $8 \times 3 = 4 \times \square$

For question 4, fill in the box and the circle with the correct number. Explain how you know that the number you put in the box and the number that you put in the circle is correct.

4) $22 + 18 = \square + 12 = \circ$

In questions 5–10, determine if each statement is true or false. Explain how you know.

- 5) $8 = 5 + 3$
- 6) $5 + \square = \square + 5$
- 7) $24 + 13 = 21 + 16$
- 8) $14 \times 8 = 7 \times 16$
- 9) $12 \times 46 = 24 \times 23$
- 10) $3 + 7 = 10 + 5 = 15 + 2 = 17$

Note that Questions 4 and 10 are Essential Questions 2 and 1 respectively from Section 3.

Throughout this unit of study you should be reflecting on your own instruction. That is, you should be looking to make connections between various ideas presented in the unit and constantly thinking about how to present those ideas to students. For example, one way students might explore question 3 above that will facilitate relational thinking (see Exercise 6.2) is by using a model.

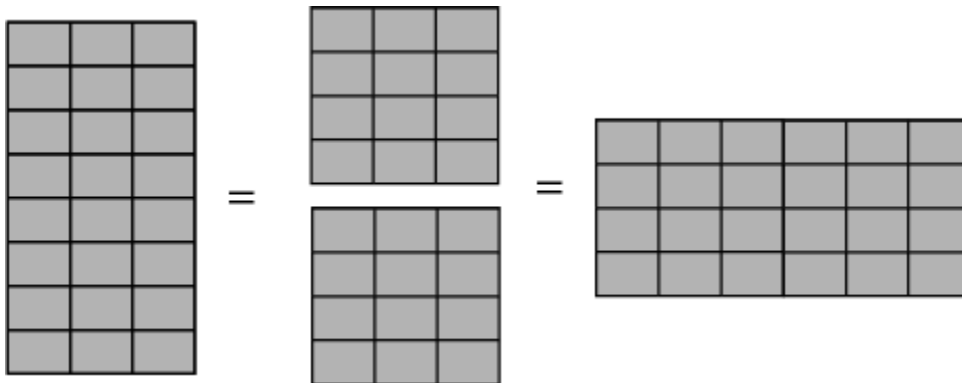


Table 6.1 – Benchmarks of Equality

Benchmark	Description
<i>Benchmark 1</i>	Questions aimed at Benchmark 1 are those that seek to elicit whether or not students have developed an understanding of the equal sign in a relational sense (i.e., “balance” vs. “here comes the answer”).
<i>Benchmark 2</i>	Questions aimed at Benchmark 2 are those that seek to elicit whether or not students have developed an understanding of alternative formats involving equality (e.g., the “answer” can precede the equal sign. See the discussion following Example 5.3).
<i>Benchmark 3</i>	Questions aimed at Benchmark 3 are those that seek to elicit whether or not students have developed an understanding of the equal sign in a relational sense and compare the expressions on both sides of the equal sign by calculating each side.
<i>Benchmark 4</i>	Questions aimed at Benchmark 4 are those that seek to elicit whether or not students have developed an understanding of the equal sign in a relational sense and use relational thinking to solve problems involving equality.

Before we see some examples of items that elicit information about students’ thinking on each of these benchmarks, along with some student work, let’s revisit Essential Question 13 and make certain that we have a clear understanding of what we mean by relational thinking.

Exercise 6.2 Take some time to revisit your response to Essential Question 13 before reading on.

Essential Question 13 Explain what is meant by relational thinking as it pertains to students solving problems involving equality like Essential Question 3. Describe some various strategies to engage students in relational thinking.

We will describe what we mean by relational thinking through two examples. Consider the following examples along with Troy's response and Nicole's response.

Fill in the box with the correct number. Explain how you know that the number you put in the box is correct.

$$47 + 21 = 45 + \square$$

Troy's work

2) $47 + 21 = 45 + \square$

45 is 2 less than 47
23 would have to be 2 more

Nicole's work

2) $47 + 21 = 45 + \square$

68 68

I added 47 + 21 and got 68 as an answer. Then I added 23 to 45 and got 68 too.

While both Troy and Nicole show evidence of understanding that the equal sign is a relational symbol (see Benchmark 1 in Table 6.1), Troy's response shows that he is able to see a relationship between the numbers in the problem as well. This relationship allows Troy to simplify the calculations used to solve the problem. While we do not know if Nicole sees this same type of relationship, her work shows that she calculated the value on each side of the equal sign to find the number that belongs in the box. (See Benchmark 3 in Table 6.1.) The type of relational reasoning that Troy used will be very useful in an algebra course.

Example 6.1 – Strategies to Engage Students in Relational Thinking

The following strategies (Examples 6.1a, 6.1b, and 6.1c) are summaries of how three teachers guided students to engage in relational thinking. Actual transcripts of the conversations appear in Carpenter, Franke, & Levi, 2003.

Example 6.1a – Guiding a second grade student to relational thinking (Adapted from Carpenter, Franke, & Levi, 2003)

Emma is a second grade student working with her teacher on solving number sentences. Emma has a set of base-ten blocks to help her make calculations and to see relationships. Throughout the conversation, Emma’s teacher consistently asks Emma what she is thinking and how she is determining her answers. This helps Emma build and construct knowledge for herself while allowing Emma’s teacher to adjust her questioning to guide Emma to relational thinking.

Emma’s teacher begins by asking Emma to find the number that belongs in the box in the statement below.

$$7 + 6 = \uparrow + 5$$

Emma quickly calculates $7 + 6$ and finds the number to add to 5 to get 13. Realizing that Emma may never need to use relational thinking when the computations are fairly easy, Emma’s teacher gives Emma a problem with larger numbers, but one that easily allows for relational thinking.

$$43 + 28 = \uparrow + 42$$

Emma still solves this problem by calculating the sum on the left-hand side (which she does quite easily using mental mathematics) and finding the number that she adds to 42 to get 71. When asked whether or not there is an easier way to solve the problem, Emma responds with a no. Adjusting again, Emma’s teacher asks Emma to find the number that belongs in the box below.

$$15 + 16 = 15 + \uparrow$$

Emma quickly solves the problem and explains how she knows so quickly that 16 is the value for \uparrow . Emma’s teacher then gives Emma the following problem.

$$28 + 32 = 27 + \uparrow$$

Emma reverts back to calculating the value for the left-hand side. Emma's teacher asks Emma if she sees a relationship between this problem and $15 + 16 = 15 + 1$. Emma still does not see how to use relational thinking to solve the problem. However, while Emma is using her base-ten blocks to calculate the sum of 28 and 32, she begins to see the relationship and hesitates. Emma's teacher asks what she is thinking and Emma begins to explain why she thinks the answer should be thirty-something and eventually explains that the answer should be 33 since 27 is 1 less than 28 and $32 + 1 = 33$. Emma then applies this type of thinking to other problems.

Exercise 6.3 Emma's teacher asked Emma a key question stands that helped Emma make the transition to relational thinking. What question was this? Explain.

Example 6.1b – Class Discussions to Facilitate Relational Thinking (Adapted from Carpenter, Franke, & Levi, 2003)

The following is a summary of how a fourth grade teacher used number sentences and class discussions to guide students to relational thinking. One of the most important aspects of this classroom is that the teacher has set up an environment where students listen to each other, discuss ideas, are expected to explain their thinking, and share a variety of different approaches to solving problems. Additionally, the teacher shows the students that she values solutions that involve more than just calculations.

Ms. K asked her fourth grade class a series of true/false questions about number sentences throughout the year. Her goals were to challenge students' understandings of equality and help students think flexibly about operations. She began with a series of relatively simple questions to get students acquainted with true/false questions dealing with number sentences. She asked whether or not the following statement was true or false.

$$12 - 9 = 3$$

Students provided several explanations as to why the above number sentence is true including explanations where 10 was subtracted from 12 to get 2 and then 1 was added to 2 to get 3. Notice that this explanation makes use of many of the properties in the previous section that needed careful development. Students using this type of reasoning are showing an informal understanding of many properties of number operations.

Ms. K continued to provide similar examples, some of which were true and some of which were false. She then asked if the following was true or false.

$$58 + 76 = 354$$

Some students quickly realized that this was false and upon being asked to explain how they knew so quickly without calculating, students responded by saying that both addends were less than 100 so the sum must be less than 200. Ms. K then provided some statements like the following.

$$34 + 18 - 18 = 34$$

$$432 + 654 - 654 = 424$$

Students were able to justify why the first statement was true; some by indicating that you add something and then just take the same something away and others by showing this concept using base-ten blocks. And, students were able to justify why the second statement was false by indicating that it was similar to the previous statement and the answer should be 432.

Subsequently, Ms. K asked students whether or not the following statement was true.

$$48 + 63 - 62 = 49$$

At first, some students thought that the calculations needed to be made. But, after some discussion, students were able to realize that this statement was similar to the previous ones and were able to answer similar questions with larger numbers without making all calculations (e.g., using reasoning such as you take away one less than what you added so the answer needs to be one more).

Later in the year, Ms. K revisited these ideas. Students were encouraged to think about the relationships between the numbers and to look at statements about equality in a relational way rather than just a series of calculations. When asked what number c has to be to make $12 + 9 = 10 + 8 + c$ true, students were able to use relational reasoning (e.g., c must be 3 since 10 is two less than 12 and 8 is one less than 1 and $2 + 1 = 3$). Ms. K was then able to increase the magnitude of the numbers in the equality statements and also guide students to solving similar types of problems using subtraction (e.g., $46 + 28 = 27 + 50 - p$). Students were able to begin to understand the properties of equality described in Table 5.6. For example, in the above subtraction problem, students were able to decompose 50 into 46 and 4, and reasoned that they didn't have to worry about the 46 on both sides (essentially using the Subtraction Property of Equality).

The type of reasoning that Ms. K's students showed is essential to success in algebra. For example, in algebra, when students try to solve problems like $3x + 21 = 40$, students need to be able to reason why 21 can be subtracted from both sides. However, as researchers indicate, students who see the equal sign as a symbol to do something tend to try to memorize a series of steps that are not embedded in understanding. Hence, these students tend to use the steps incorrectly and are not able to work flexibly among them (Falkner, Levi, & Carpenter, 1999).

**Example 6.1c – Student Generated Number Sentences
(Adapted from Carpenter, Franke, & Levi, 2003)**

Another useful way to engage students in relational thinking is to have them create their own true/false number sentences and share them with their classmates. This type of technique can be used throughout mathematics instruction and is often an enlightening technique. Students enjoy trying to challenge their classmates, and this type of activity allows the teacher to quickly assess whether or not students have developed a deep understanding of the concepts that they are studying.

Exercise 6.4 Create some examples of number sentences that could encourage relational thinking. Make certain to create examples using addition, subtraction, multiplication, and division (or even using more than one operation). Also, make certain that some of the example have large enough numbers to discourage calculation. Finally, make certain that some numbers sentences are true while others are false.

Exercise 6.5 Revisit Exercise 6.1 and sort questions 1–10 according to which Benchmark(s) of Equality from Table 6.1 you believe a particular question was aimed at. Be sure to justify your choices. Also, determine which questions you solved by using relational thinking.

Exercise 6.6 Look at the student work below on some of the problems from Exercise 6.1 and answer the questions that follow.

Mike's work

4) $22 + 18 = \boxed{40} + 12 = \textcircled{52}$

$$\begin{array}{r} 22 \\ + 18 \\ \hline 40 \end{array}$$

I'm pretty sure the answer in the box is 40 because $22 + 18 = 40$

AND then in the circle $40 + 12$ equals 52

Alicia's work

8) $14 \times 8 = 7 \times 16$

100 false

$14 \times 8 = 112$
and $7 \times 16 = 112$

Anthony's work

10) $3 + 7 = 10 + 5 = 15 + 2 = 17$ True

✓ ✓ ✓

- a) What misconception about the equal sign does Mike’s response exhibit?
- b) Many students who exhibited relational thinking on problems such as number 2 from Exercise 6.1, solve number 4 from Exercise 6.1 as illustrated by Mike’s solution. That is, even though students showed some understanding of equality as a relation, once a number of equal signs were strung together with some unknowns, students seemed to revert back to treating the equal sign as an operator or symbol for writing down the answer. Describe some strategies to help students use relational thinking to solve problems like number 4 from Exercise 6.1.
- c) Alicia’s work seems to indicate that she tried to calculate the value for the expressions on both sides of the equal sign, but seemed to make a computation error leading her to state that the equality statement was false. Explain how to guide Alicia to using relational thinking to solve this problem. You may also want to think about how to help Alicia determine if her calculations are reasonable.
- d) What do the check marks in Anthony’s work seem to indicate?
- e) Anthony’s problem is also Essential Question 1. Revisit your answer to that problem and explain why the number sentence is false.

Example 6.2 In addition to creating number sentences that encourage relational thinking, as in Exercise 6.4, while attending to using various operations, it is important to attend to other features such as the position of the unknowns, the use of unknown versus variables, the number of addends, and the number of operations. Let’s begin this discussion by revisiting Essential Question 14 from Section 3.

Essential Question 14 Is there any difference between an unknown and a variable? Explain.

While the words “unknown” and “variable” are often used interchangeably, it can be helpful to identify when a variable is being used as an unknown, along with the different ways that variables are used (see Table 6.4). An unknown can refer to a situation involving a conditional equality. (See Table 5.1.) That is, a value that is to be found. For example, what number should \square be replaced with in $7 + \square = 12$ to make the number sentence true. Whereas, sometimes a variable represents a number of values. For example, if Ray makes \$8 per hour mowing lawns, the expression $8x$ represents the amount of money Ray makes if he mows lawns for x hours. Notice that the amount of money Ray makes depends on the number of hours he works. When introducing number sentences to students involving variables, it is important to engage students in conversations that elicit their preconceptions on how notation is used. Essential Question 15 illustrates this point.

Essential Question 15 Find all values for the symbols in the following statement that make the equality true.

$$\square + \square = 12$$

Find all values for the symbols in the following statement that make the equality true.

$$\square + \Delta = 12$$

Describe how the above questions are similar or different.

There may be some confusion as to whether or not different numbers can be placed in the boxes in the first number sentence and whether or not the same number can be used in the box as in the triangle in the second number sentence. Take a moment to revisit how you responded to this question.

Certain conventions have been agreed upon by the mathematics community for these cases. In the first case, the same number must be used in both boxes. That is, whenever the same symbol appears more than one time in a number sentence, it must be replaced by the same value. When students learn this concept they tend to over generalize and assume that when different symbols are used in a number sentence they can not be replaced by the same value (Carpenter, Franke, & Levi, 2003). However, in the second case, when two different symbols are used, they may or may not represent the same value (e.g., one solution to the second number sentence is to replace the box with 6 and the triangle with 6, whereas, a second solution is to replace the box with 1 and the triangle with 11). Also, notice that we didn't specify the set of numbers from which the solutions can come from. In the second number sentence, if we are working within the set of whole numbers there are a finite number of solutions (take some time to verify this); whereas, if we are working within the set of integers there are an infinite number of solutions (Whatever value is chosen for the triangle, we can make a true number sentence by choosing $12 - \Delta$ for the value of the box).

While these conventions may not be immediately clear, we want to be able to represent different mathematical situations with these different number sentences. As an example, the first number sentence models algebraic situations that are represented by $2x = 12$. For instance, Troy has two packages that have a combined weight of 12 pounds. Each package weighs the same. What is the weight of one package? Notice in this case x is a variable that is being used as an unknown. We want the second number sentence to model algebraic situations represented by $x + y = 12$. As an example, Alisa has 12 pieces of fruit. She has some apples and some oranges. What are all the different possible numbers of apples and oranges she could have? Notice in this case that x and y variables being used in a way where the value of one variable depends upon the value of the other variable. And, it is certainly the case that Alisa could have 6 apples and 6 oranges (i.e., the different symbols can be replaced by the same value to produce a true number sentence). Additionally, this relationship can be modeled by the function $y = 12 - x$ or $\square = 12 - \Delta$.

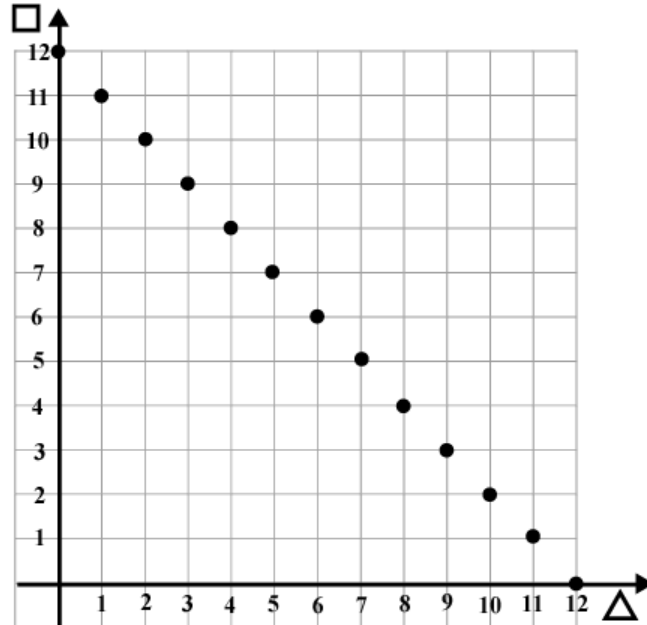
In situations where the value of one variable depends upon the value of another variable, students should be encouraged to make function tables similar to Table 6.2 to organize their work. A good activity to try with students is the guess my rule activity. Essentially, you ask students to make up a rule. Then, you give students various numbers and ask them to use their rules to perform some operations on those numbers to generate new numbers. Finally, you or other students try to guess the various rules (Adapted from W.W. Sawyer’s Guess My Rule Game as described in Kieran, 1991). In the case of our example, the rule would be to take whatever number is given and subtract it from 12.

Table 6.2 – Using a Table to Represent a Functional Relationship

Δ	\square
0	12
1	11
2	10
3	9
4	8

Another way to have students generate a table like this without using symbols is to say that you are thinking of two numbers that add to 12 and ask what those numbers could be.

The equation $\square + \Delta = 12$ should look familiar from teaching number facts. The following graphical representation shows the whole number values for \square and Δ that make the number sentence true. Notice how this is a linear relationship.



Since there isn't always a distinction made between the terms variable and unknown, it is useful to think about the common ways the term variable is used as shown in Table 6.3. Notice the connections between the different ways variables are used to the different ways the equal sign is used (see Table 5.1).

Table 6.3 – Three Common Ways the Term Variable is Used in K-12 Mathematics (Adapted from Cuevas, Yeatts, & House, 2001)

Description	Examples	Connection to Table 5.1
<i>Variable as Unknown</i>	$8x = 3x + 25$ <p>Here, we are looking for a particular value for x that results in the values of both sides of the equation being the same. There are some values that make the equality true and others that make it false. The role of the variable is similar to a place holder.</p>	See Conditional Equality.
<i>Variables in Generalizations</i>	$a \times 0 = 0$ <p>Here the variable is used to represent any number. That is, in an identically equivalent way.</p>	See Identically Equivalent Expressions.
<i>Variables as Quantities that Vary</i>	$\square + \Delta = 12$ <p>Here the variables are used to represent a number of values, where the value of one variable depends upon the value of another variable (e.g., once a choice is made for Δ, $\square = 12 - \Delta$). This might also be described as covariation.</p>	See Definitional Equality. (Also see National Council of Teachers of Mathematics, <i>Principles and Standards for School Mathematics</i> , 2000, pg. 225.)

Exercise 6.7 Look back at Example 31.3 from the NECAP Number and Operations Support Materials shown in Section 4 on page 9. Determine how the shapes in that example are being used according to the classifications in Table 6.3. Explain.

Varying features of number sentences such as the number of unknowns, variables, addends, or operations can help students develop a deeper understanding of equality and build a solid algebraic foundation. Table 6.4 shows some examples.

Table 6.4 – Examples of Varying Features of Number Sentences

Example	Description of Various Features
$\square + \Delta = 12$	Use of variables as quantities that vary; multiple solutions; promotes discussion as to whether or not different symbols can stand for the same numbers
$18 + 12 = \Delta \times 2 + \Delta \times 3$	Multiple operations; multiple use of an unknown on the right side; promotes relational thinking; possible use of distributive property
$x + x + w = 16$	Two variables in a covariation situation with one appearing twice; there are a number of different whole number combinations making the number sentence true
$\uparrow + \uparrow + \uparrow + \uparrow = 12 + 2$	Multiple addends; each box represents the same unknown; no integer solution; promotes relational thinking
$8x = 3x + 25$	Unknown on both sides; promotes relational thinking; uses multiple operations
$\uparrow \times 4 = 20 - \uparrow$	Different position of unknowns; unknowns on both sides; uses multiple operations
$\square + \square + \square = 12$ $\square + \bullet + \blacktriangle = 12$ $\blacktriangle + \blacktriangle + \blacktriangle + \blacktriangle = 12$	Use of multiple unknowns and multiple number sentences
$\Delta \times \square = 24$	Two variables in a covariation situation; strong connection to number facts; represents an inversely proportional situation (see Rhode Island Department of Education, 2007)

Without using symbols, you might say to students that you are thinking of two numbers whose product is twenty-four and then ask them what those numbers could be.

Before reading on, take a moment to predict what the graphical representation of this function looks like. Compare your prediction to the graph following Table 6.2. Then, create a function table to show various solutions to this number sentence and use that table to create a graphical representation. Discuss what happens to the value of \square as Δ increases.

The following is a function table representing the whole numbers whose product is twenty-four (i.e., $\Delta \times \square = 24$). At the right is a graph of the function for whole number values of Δ with the curve fitting the function shown in light gray. Notice that as the value of Δ increases, the value of \square decreases.

Δ	\square
1	24
2	12
3	8
4	6
6	4
8	3
12	2
24	1

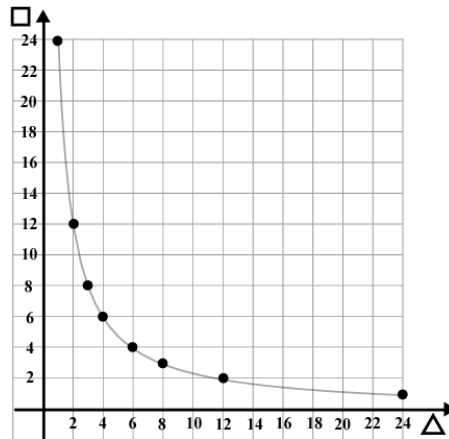
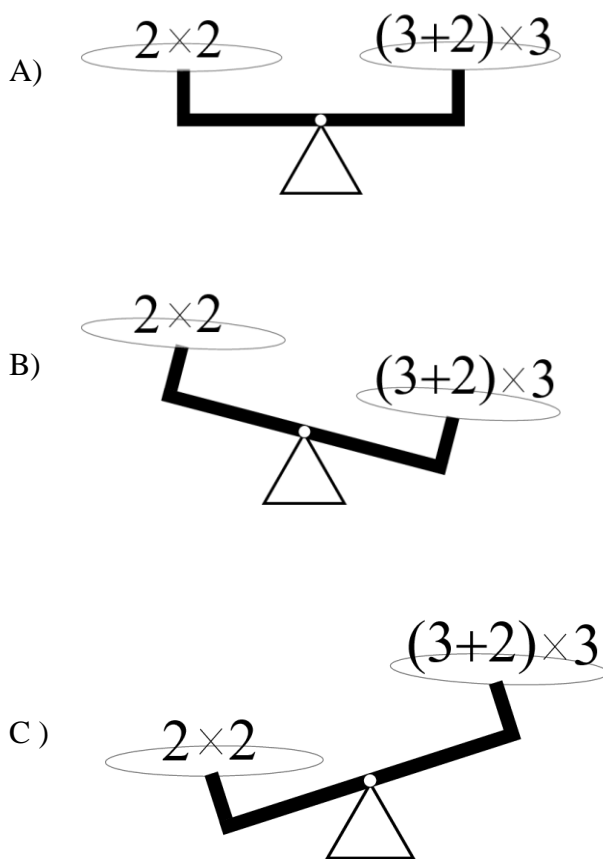


Table 6.4 only shows a few examples of number sentences that involve various features that are important for students to think about. It is important to purposely design problems that challenge students' understandings of equality.

Exercise 6.8 Look at the second and fourth number sentences in Table 6.4. Determine how students might solve each of these number sentences using relational thinking.

Exercise 6.9 Create some number sentences that pay attention to the features discussed in Tables 6.1 and 6.4 that are grade appropriate for the students that you teach. Try your number sentences out with your students and watch for relational thinking and various strategies for solving the number sentences.

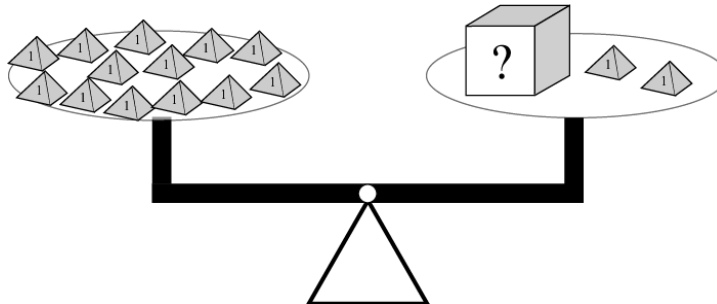
In addition to presenting number sentences in words before transitioning to a symbolic form, balance scales can be used to help students visualize equality problems. It is often beneficial to begin by helping students understand the scales. That is, helping them understand what the scales look like when they are balanced and when one side is heavier than the other. Additionally, this helps students to build an understanding of inequalities. For example, which scale models $2 \times 2 = (3 + 2) \times 3$? (Adapted from Cuevas, Yeatts, & House, 2001)



M:N&O:4:4 **Accurately solves problems involving multiple operations on whole numbers ...**

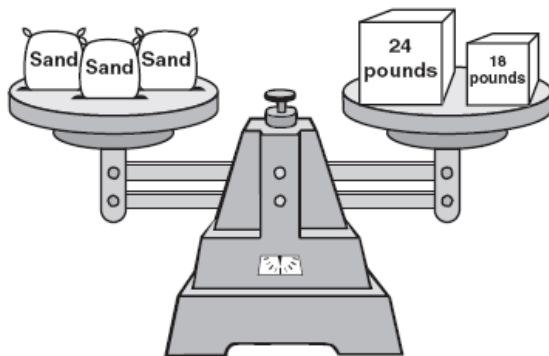
(IMPORTANT: Applies the conventions of order of operations where the left to right computations are modified only by the use of parentheses.)

Example 6.3 The following balance scale can be used to model question 1 in Exercise 6.1 ($8 + 5 = \square + 2$). Each pyramid weighs 1 pound, how much does the cube need to weigh to balance the scale?



Example 6.4 The following 2005 NECAP Released Item from Grade 6 models the number sentence $\Delta + \Delta + \Delta = 24 + 18$ or $3 \times \Delta = 24 + 18$ or $3x = 24 + 18$.

14 The scale shown below is balanced.



NECAP 2005 Released Item – Grade 6,
M:F&A:5:4

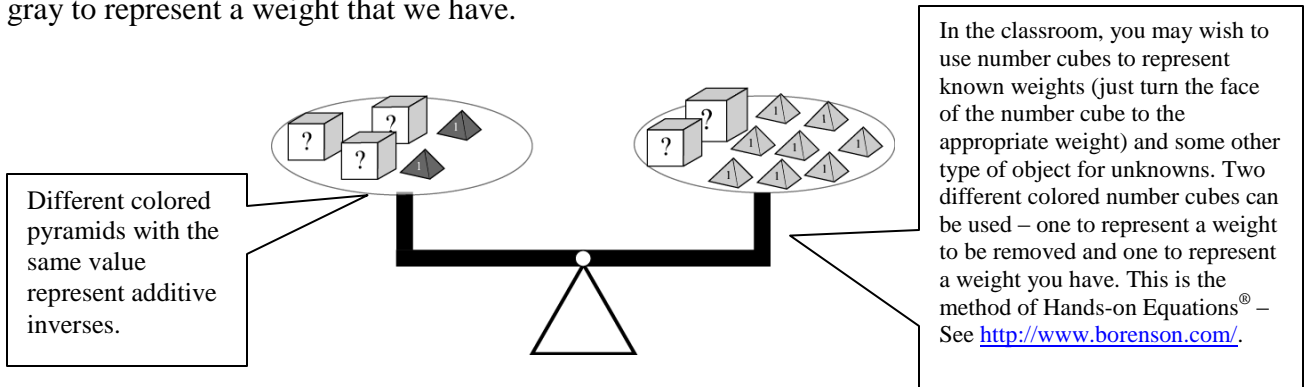
Each bag of sand weighs the same. How many pounds does one bag of sand weigh? Show your work or explain how you know.

Exercise 6.10 Describe how to use relational thinking to determine the weight of each sand bag in Example 6.4.

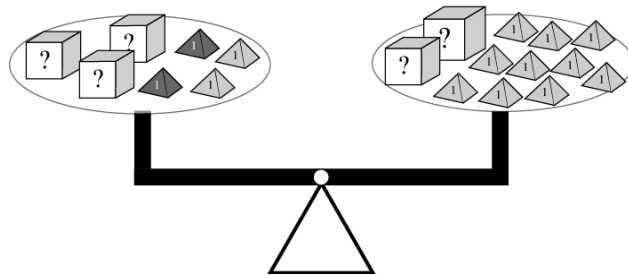
Exercise 6.11 Draw a balance scale to represent the number sentence $3x + 2 = 2x + 8$ or $\square + \square + \square + 2 = \square + \square + 8$ or $3 \times \square + 2 = 2 \times \square + 8$. Describe how a student might use the scale to solve the problem.

Example 6.5 – Number Sentences involving Subtraction and Balance Scales

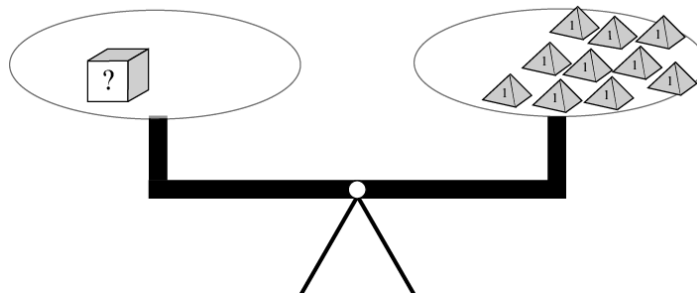
You may be wondering how to model number sentences that involve subtraction using balance scales. For example, $3x - 2 = 2x + 8$ or $3 \times \square - 2 = 2 \times \square + 8$. We just need to decide which objects represent our unknowns and which objects represent our known values. We will use cubes to represent our unknown, where we are looking for the weight of the cube to balance the scale. We will use pyramids that weigh 1 pound each to represent our known weights. Since on the left-hand side of our scale we need to remove two pounds and on the right hand side of our scale we have eight pounds, we will use two different colored pyramids – dark gray to represent a weight we need to remove and light gray to represent a weight that we have.



Since we need to remove two pounds from the left-hand side that are not shown, we will add two pounds to both sides (See the Addition Property of Equality in Table 5.6.). Symbolically this is represented as $3x - 2 + 2 = 2x + 8 + 2$.



We now use the fact that $1 - 1 = 0$ on the right-hand side to remove all the pyramids on that side. (In Section 5, we defined subtraction in terms of additive inverses. That is, $1 - 1 = 1 + (-1) = 0$.) We now see that we can remove two cubes from the left-hand side and two cubes from the right-hand side since they have the same weight. This results in the solution of our number sentence (i.e., the weight of one cube is 10 pounds).



Exercise 6.12 Create some number sentences using balance scales that are appropriate for your students.

Exercise 6.13 Explain why many students would say that the number sentence $x + y = y + z$ can never be true. Give an example for when this number sentence is true.

Exercise 6.14 Explore solving linear equations using balance scales by going to the National Library of Virtual Manipulatives, <http://nlvm.usu.edu>, and selecting Virtual Library followed by clicking on Algebra under the 9–12 heading.

As students begin to build a deeper understanding of equality, it is important that they explore many of the properties discussed in Section 5. This is the focus of Section 7.

Section 7: Students' Understandings of Properties and Operations

As students begin to build a deeper understanding of equality, it is important that they explore many of the properties discussed in Section 5. Many of these properties will be posed as conjectures as students work through equality statements (see Carpenter, Franke, & Levi, 2003). Students should be encouraged to make conjectures and explore those conjectures. These conjectures will help students build upon their preconceptions and begin to bridge procedural fluency with conceptual understanding while building new knowledge (see Table 1.1). We will explore some of these ideas in the next series of exercises and examples. These are all properties that were discussed in Section 5. We will explore these from a teaching perspective. You are encouraged to apply this type of perspective to the properties of Section 5 that we do not discuss here. Notice that most of the properties that we discussed in Section 5 provide examples of using the equal sign in an identically equivalent way (see Table 5.1). Before introducing students to properties that are true no matter what the value of the variable is, it is useful to have them try to come up with some number sentences that are always true.

Exercise 7.1 Create some examples of open number sentences that are always true no matter what numbers you put in for the variables.

Exercises similar to Exercise 7.1 lead to many student conjectures. While some of these conjectures will be false, students should be encouraged to explore all conjectures. Students should be encouraged to provide justification for conjectures which are true and support their justifications with examples. And, students should be encouraged to provide counterexamples for conjectures that they determine to be false. Table 7.1 lists some typical conjectures and describes the conjectures in symbols and words. The table also links the conjectures to properties discussed in Section 5 and provides some examples or counterexamples for the conjectures.

Table 7.1 – Some Conjectures to Explore

Conjecture in Words	Conjecture in Symbols	Some Examples or Counterexamples	Connection to Section 5
If you subtract any number from itself you get 0.	$a - a = 0$	$5 - 5 = 0$ $25 - 25 = 0$ $0 - 0 = 0$	See the Additive Inverse Property in Table 5.4.
If you add any number to 0 you end up with the number that you started with.	$a + 0 = a$	$8 + 0 = 8$ $25 + 0 = 25$ $0 + 0 = 0$	See the Additive Identity Property in Table 5.4.
If you multiply any number by 1 you end up with the number that you started with.	$a \cdot 1 = a$	$8 \cdot 1 = 8$ $25 \cdot 1 = 25$ $0 \cdot 1 = 0$	See the Multiplicative Identity Property in Table 5.4.
If you divide any number by itself you get 1.	$a \div a = 1$	$8 \div 8 = 1$ $25 \div 25 = 1$ $0 \div 0$ is undefined – Why?	See the Multiplicative Inverse Property in Table 5.4.
Any number times zero is zero.	$a \times 0 = 0$	$8 \cdot 0 = 0$ $25 \cdot 0 = 0$ $0 \cdot 0 = 0$	See Table 5.7.
It doesn't matter what order you add two numbers.	$a + b = b + a$	$5 + 3 = 3 + 5$ $18 + 53 = 53 + 18$ $124 + 456 = 456 + 124$	See the Commutative Property for Addition in Table 5.4.
It doesn't matter what order you multiply two numbers.	$a \times b = b \times a$	$5 \times 3 = 3 \times 5$ $18 \times 53 = 53 \times 18$ $124 \times 456 = 456 \times 124$	See the Commutative Property for Multiplication in Table 5.4.
It doesn't matter what order you subtract two numbers.	$a - b = b - a$	This conjecture is false. A counterexample is $8 - 2 \neq 2 - 8$.	See previous column.
It doesn't matter what order you divide two numbers.	$a \div b = b \div a$	This conjecture is false. A counterexample is $24 \div 6 \neq 6 \div 24$.	See previous column.

Exercise 7.2 Determine some other typical conjectures (i.e., ones not listed in Table 7.1) that students might suggest. Write the conjectures in both symbols and words.

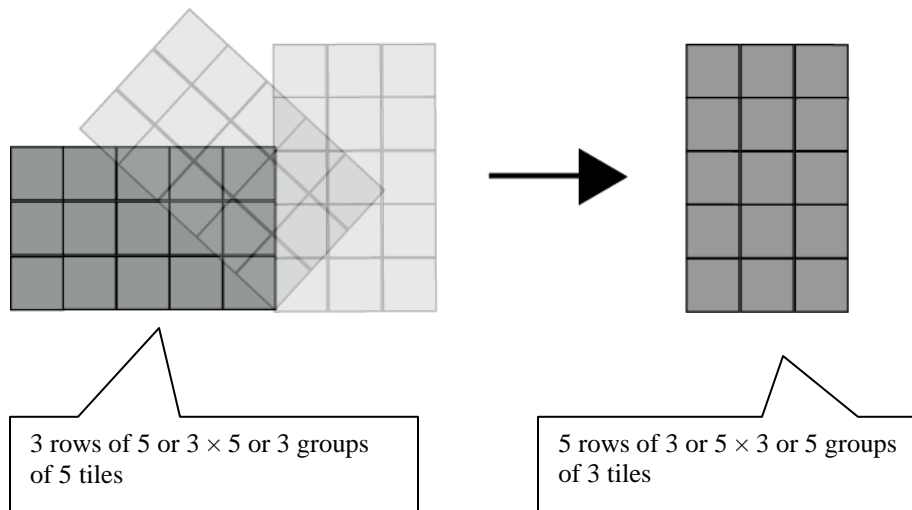
Exercise 7.3 Explain why division by 0 is undefined.

Example 7.1 – Understanding the Commutative Property of Multiplication using an Array

Take a moment to revisit Essential Question 9 from Section 3.

Essential Question 9 Describe, in your own words, the Commutative Property of Multiplication. Support your description with examples, and draw a model to illustrate the property.

The following illustration shows how students might use manipulatives to help understand the Commutative Property of Multiplication. The illustration shows that $3 \times 5 = 5 \times 3$. Recall the connection made in Section 5 between multiplying two numbers and area. This example also illustrates that a 3 by 5 rectangle has the same area as a 5 by 3 rectangle. We can just rotate the 3 by 5 rectangle ninety degrees about its lower right-hand corner to obtain the 5 by 3 rectangle.



Students should explore many numeric examples to feel comfortable with this property and should be able to connect this property to words (i.e., it doesn't matter what order you multiply in). Subsequently, students can be guided to connecting the words to symbols (i.e., $a \times b = b \times a$).

Exercise 7.4 Revisit Essential Question 8 from Section 3. Describe how students might justify the Commutative Property of Addition by using manipulatives, along with the words that they might use to describe the property.

There are some conjectures that students do not often come up with on their own; particularly, conjectures involving three numbers such as the distributive properties or associative properties. Researchers have found that it is uncommon that students work with adding more than two numbers. Therefore, students often don't think about how they can group the numbers (Carpenter, Franke, & Levi, 2003). Even though students might not intuitively see how to build and construct models for these problems we can help guide them to these constructions. Therefore, we will spend some time exploring models for these properties and will also revisit some of the essential questions from Section 3.

Example 7.2 – Understanding the Distributive Properties using a Model

Take a moment to revisit Essential Question 12 from Section 3.

Essential Question 12 Describe, in your own words, the Distributive Properties. Support your description with examples, and draw a model to illustrate the properties.

The following illustration shows how manipulatives may be used to help understand one of the distributive properties.

$$3 \times (4 + 5) = 3 \times 4 + 3 \times 5$$

3 groups of 9 can be broken into 3 groups of 4 plus 3 groups of 5. In words, this property shows that when you multiply a number n (in this case 3) by the sum of two numbers (in this case 4 and 5) you can either find the sum of the two numbers first and then multiply your number n by this sum or you can multiply n by each of the addends and then find the sum of these results.

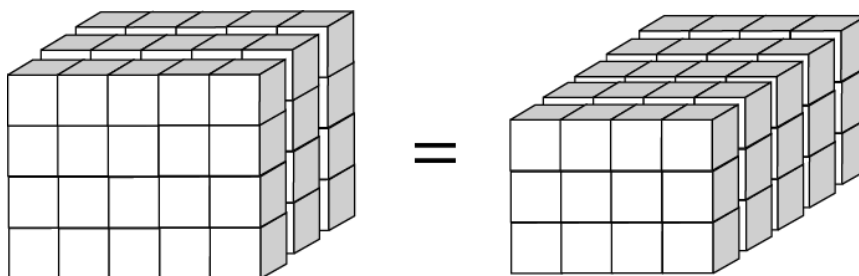
Example 7.3 – Understanding Associative for Multiplication using a Model

Take a moment to revisit Essential Question 11 from Section 3.

Essential Question 11 Describe, in your own words, the Associative Property of Multiplication. Support your description with examples, and draw a model to illustrate the property.

Just as multiplication of two numbers can be thought of as representing the area of a rectangle (see page 18), multiplication of three numbers can be thought of as the volume of a rectangular prism. Hence we can use a three-dimensional model to show the Associative Property of Multiplication. While students don't encounter volume of rectangular prism until M:G&M:5:6, they can use different colored tiles or unifix cubes to represent the Associative Property of Addition (see Exercise 7.5).

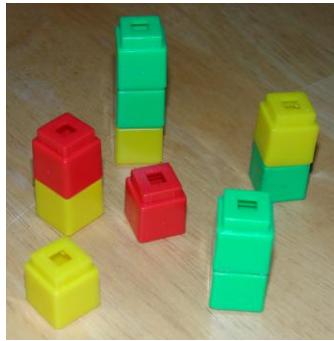
M:G&M:5:6 **Demonstrates conceptual understanding of perimeter** of polygons, and **the area of** rectangles or right triangles through models, manipulatives, or formulas, the area of polygons or irregular figures on grids, and **volume of** rectangular prisms (cubes) using a variety of models, manipulatives, or formulas. Expresses all measures using appropriate units.



$$3 \times (4 \times 5) = (3 \times 4) \times 5$$

When multiplying three numbers, it doesn't matter if we start by multiplying the last pair of numbers or the first pair of numbers. Notice how the resulting rectangular prisms have the same volume.

Exercise 7.5 Show how to use unifix cubes to model the Associative Property of Addition.



Exercise 7.6 Revisit your solutions to Exercises 5.15, 5.16, and 5.17.

Exercise 7.7 The properties discussed in this section are important strategies used in mental mathematics. Mental mathematics will be explored in subsequent units of study. However, you may wish to explore the topic of numerical literacy by reading *Future Basics: Developing Numerical Power – A Monograph of the National Council of Supervisors of Mathematics*. Most likely, the monograph can be obtained from your state department of education. The references is also cited in Section 11.

Now that we have a better understanding of equality, along with some research connected to students' understandings of equality, Section 8 will examine some NECAP released items and give you the opportunity to answer some questions related to these items that are based on the ideas presented in this unit of study.

Section 8: Examining NECAP Released Items and Student Work

This section contains items released from the New England Common Assessment Program, along with some sample student work (for constructed response items) and statistics on the items. Each released item is mapped to a primary Grade-Level Expectation as indicated by the codes next to the items. These Grade-Level Expectations can be located in Section 2. This section will allow you to see how students approach some problems that connect to equality. Each NECAP item will be followed by a series of questions that will allow you to view the item within the contexts of this unit of study. Furthermore, the questions following each released item will give you ideas for questions that you can use to challenge students' understandings of equality. The items and students' responses in this section are reprinted with permission from the New England Common Assessment Program (NECAP).

Exercise 8.1 Read through the following released NECAP item and student responses and answer the questions that follow.

- 13 Look at this number sentence.

$$\triangle + \triangle + \triangle = 7 + 2$$

NECAP 2005 Released Item –
Grade 4, M:F&A:3:4

Each triangle has the same value.

What is the value of each triangle?

Item Statistics*
(percent of students earning each score point)
0 – 37%
1 – 63%

*statistics from NECAP pilot assessment, $n = 1599$

Kris's work

9

Look at this number sentence.

$$\begin{array}{cccc} 3 & 3 & 3 & 9 \\ \Delta & + & \Delta & + & \Delta & = & 7 & + & 2 \end{array}$$

Each triangle has the same value.

What is the value of each triangle?

9 First I add $7+2=$
 then I thought what $\times 3 = 9$?
 I knew $3 \times 3 = 9$ so that is your
 answer.

- What might we be able to learn from Kris's response relative to the benchmarks of equality shown in Table 6.1? Explain.
- Explain how to solve this problem using the ideas in Benchmark 3 from Table 6.1.
- Explain how to solve this problem using the ideas in Benchmark 4 from Table 6.1.
- What difficulties might this problem pose for students who haven't had much experience with questions written to Benchmark 2 from Table 6.1?
- Describe a way to change the problem to specifically assess Benchmark 1 from Table 6.1.
- Look at Alisa's response. Which benchmark(s) of equality does Alisa show an understanding of? Explain.

Exercise 8.2 Read through the following released NECAP item and answer the questions that follow.

- 3 Look at this number sentence.

$$14 + 14 = 12 + \square$$

What number makes this number sentence true?

- A. 16
- B. 18
- C. 28
- D. 40

NECAP Practice Test – Grade 3,
M:N&O:2:1

Item Statistics*
(percent of students choosing each option)

A – 35%

B – 7%

C – 32%

D – 23%

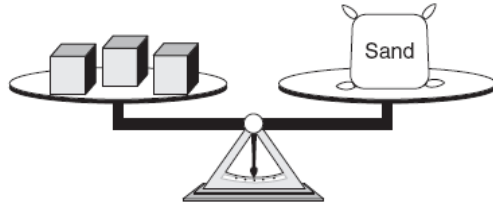
* statistics from 2004 NECAP pilot
assessment, $n = 1599$

- a) Examine each of the answer options and determine how a student might arrive at each option.
- b) Which answer options are designed to assess whether or not students have met Benchmark 1 from Table 6.1? Explain.
- c) Based on the statistics from this item, what percent of students show that they most likely view the equal sign as an operator rather than in a relational way? Explain.
- d) Explain how to solve this problem using relational thinking.
- e) Create a balance scale to represent this problem, and show how to use the balance scale to solve the problem using relational thinking.

Exercise 8.3 Read through the following released NECAP item and student responses and answer the questions that follow.

- 12 The scale shown below is balanced.

NECAP 2005 Released Item –
Grade 5, M:F&A:4:4



The bag of sand weighs 18 pounds. Each of the cubes has the same weight. How many pounds does one cube weigh?

Item Statistics*
(percent of students earning each score point)
0 – 17%
1 – 83%

* statistics from 2004 NECAP pilot assessment, $n = 1349$

Karen's response

They all together way 54 pounds.

- Explain how Karen most likely determined her response to the problem.
- What additional questions might you ask Karen to help her get to the correct answer?
- Write a number sentence to model this problem.

Exercise 8.4 Read through the following released NECAP item and student responses and answer the questions that follow.

- 16 a. Write a number in the box to make this number sentence true.

$$3 + 6 = \square + 5$$

NECAP 2006 Released Item –
Grade 3, M:F&A:2:4

- b. Write the numbers 8, 6, 2, and 4 in the boxes to make this number sentence true. Use each number only once.

$$\square + \square = \square + \square$$

Item Statistics*
(percent of students earning each score point)
0 – 62%
1 – 11%
2 – 27%

* statistics from 2006 NECAP Assessment from NH

Ian's
response

- 16 a. Write a number in the box to make this number sentence true.

$$3 + 6 = \square + 5$$

4 makes
this number
sentence
true

- b. Write the numbers 8, 6, 2, and 4 in the boxes to make this number sentence true. Use each number only once.

$$6 + 4 = 8 + 2$$

Lindsay's response

- 16 a. Write a number in the box to make this number sentence true.

$$3 + 6 = \boxed{9} + 5$$

- b. Write the numbers 8, 6, 2, and 4 in the boxes to make this number sentence true. Use each number only once.

$$\boxed{2} + \boxed{4} = \boxed{6} + \boxed{8}$$

Mark's response

- 16 a. Write a number in the box to make this number sentence true.

$$3 + 6 = \boxed{4} + 5$$

- b. Write the numbers 8, 6, 2, and 4 in the boxes to make this number sentence true. Use each number only once.

$$\boxed{8} + \boxed{6} = \boxed{2} + \boxed{4}$$

a) Each part of the item was worth one point (i.e., part a – one point for the correct answer; part b – one point for a correct answer). Determine Ian's, Lindsay's, and Mark's scores.

b) Create a balance scale representation for part a of this item.

c) Determine all possible true number sentences for part b using the numbers 8, 6, 2, and 4. Explain how relational thinking can help find all the possible true number sentences.

Exercise 8.5 Read through the following released NECAP items and determine some of varying features of the number sentences similar to those described in Table 6.4 (e.g., the position of the unknown, the operations used, the number of addends, and design that may facilitate relational thinking).

- 12 Look at this number sentence.

$$\square - 5 = 9$$

What number makes this number sentence true?

NECAP 2005 Released
Item – Grade 3,
M:F&A:2:4

- 5 Look at this number sentence.

$$12 - \square = 9 - 4$$

What number makes this number sentence true?

NECAP Practice Test
Item – Grade 3,
M:F&A:2:4

- 5 Look at this number sentence.

$$33 + 53 = \square + 41$$

What number makes the number sentence true?

NECAP Practice Test
Item – Grade 4,
M:F&A:3:4



- 15 Look at this number sentence.

$$2 \times 4 = \square \times 8$$

What number makes the number sentence true? Show your work or explain how you know.

NECAP 2006 Released Item –
Grade 4, M:F&A:3:4

- 8 Look at this number sentence.

$$12 - 7 = \square - 3$$

What number makes this number sentence true?

- A. 2
- B. 5
- C. 7
- D. 8

NECAP 2006 Released Item –
Grade 3, M:F&A:2:4

Exercise 8.6 Read through the following released NECAP item and student responses and answer the questions that follow.

- 15 In these number sentences, each star represents the same number and the heart represents a different number.

$$\star + \star + \star = 12$$

$$\heartsuit - 2 = \star$$

NECAP 2006 Released Item –
Grade 5, M:F&A:4:4

- a. What number does the star represent? Show your work or explain how you know.
- b. What is the value of one star plus two hearts? Show your work or explain how you know.

Item Statistics*
(percent of students earning each score point)

0	– 11%
1	– 6%
2	– 21%
3	– 26%
4	– 36%

* statistics from 2006 NECAP Assessment from NH

Ava's response

the star represents 2.46=12 because 2+4+6=12

$$\star 2 + \star 4 + \star 6 = 12$$

the value of 2 hearts + 1 star = 12

$$\frac{-12}{12}$$

Ron's response

A. $4 + 4 + 4 = 12$

B. $4 + \heartsuit + \heartsuit = 18$

6 6

Quinn's response

A. ④ $4 + 4 + 4 = 12$. $4 \times 3 = 12$.

B. ⑩ IF 1 star = 4, then the answer is 4. Now the problem looks like this.
 $? - 2 = 4$. $b - 2 = 4$. So if the heart = b, then $4 + b + b = 16$.

a) Each part of the item was worth two points (i.e., part a – two points for correct answer with appropriate explanation or one point for correct answer with incomplete or no explanation or one point for incorrect answer with correct strategy; part b – scored the same as part a). Determine Ava's, Ron's, and Quinn's scores.

b) After solving part a, Quinn indicates that the “answer” to part b is 4. While we can’t be certain without asking Quinn what she was thinking, Quinn’s response possibly shows that she believes that the answer must come after the equal sign. How can you amend the problem slightly to see if Quinn has this misconception?

c) In part a, Ava doesn’t seem to understand that the stars represent the same unknown value. Thus, without intervention, she is likely to also over-generalize information from this problem – assuming that different symbols must always stand for different numbers. What are some additional questions that you might ask Ava to help her understand variables?

Exercise 8.7 There are many other released items that involve understanding equality. Examine the NECAP Practice Tests and Released NECAP items from 2005 and 2006 and find other examples of items and student work that connect to equality.

Section 9: Summary

Understanding equality is deeper than performing computations. It requires an understanding of a relation and the ability to treat the symbol “=” as a relational symbol rather than an operator. Students come to the classroom with many preconceptions about equality. We need to challenge students’ understandings of equality to help foster algebraic thinking.

Exercise 9.1 Examine your instructional materials and district curriculum and determine if they provide opportunities to challenge students’ understandings of equality (by varying features of number sentences such as the position of the unknowns, the number of addends, the magnitude of the numbers involved), along with opportunities for students to develop and test conjectures related to properties of equality.

Exercise 9.2 Examine your instructional materials and district curriculum and determine if they address various ways that the equal sign is used and various ways that variables are used.

Exercise 9.3 Examine your instructional materials and district curriculum and determine if they provide ample opportunities for students to use a variety of representations (e.g., tables, graphs, symbols) when working with number sentences (e.g., $\square + \Delta = 24$) and to make connections between them.

Exercise 9.4 Create a curricular sequence/unit of study for the topic of equality. The sequence should consider all three principles of how students learn. The following table can serve as a model for your work.

Concept	Description of How Concept is Introduced	Target Depth of Knowledge Levels*	Description of Activities

*You may want to review Appendix A for information on Depth of Knowledge.

This unit of study was meant to introduce you to research around equality that can have a direct impact on classroom instruction, but also deepen your content knowledge. The unit of study was intended to be used as a supplement to current curricular materials and provide examples that challenge students’ understandings of equality.

If you are looking to build upon the ideas presented in this unit of study and increase your depth of understanding of equality, you are encouraged to explore the references listed in Section 11.

Section 10: Answers to Exercises

Section 2: Connecting to the Grade-Level Expectations

Exercise 2.1: Primary GLEs given on pp. 4-5 of Section 2.1. Note: Additional GLEs deal with equality – only those that are the main focus of this unit of study are listed on pp. 4-5.

Exercise 2.2: See the Curriculum Focal Points document from NCTM to find these.

Section 3: Essential Questions

Essential Question 1: False; Explanations will vary. In this question the equal sign is being treated as an operator rather than as a relation.

Essential Question 2: $\square = 28$, $\Delta = 40$; Explanations will vary. One approach is to begin by finding the sum of the two addends on the left-hand side. Another approach is to use relational thinking (e.g., 12 is 6 less than 18, so \square is 6 more than 22).

Essential Question 3: Two common wrong answers are 13 and 15 – both found by treating the equal sign as an operator or symbol signifying to write down the answer.

Essential Question 4: Answers will vary. See Section 5.

Essential Question 5: Answers will vary.

Essential Question 6: Answers will vary. See Table 5.1

Essential Question 7: See discussion in Section 5.

Essential Question 8: See Exercise 7.4.

Essential Question 9: See Example 7.1.

Essential Question 10: See Exercise 7.5.

Essential Question 11: See Example 7.3.

Essential Question 12: See Example 7.2.

Essential Question 13: See Exercise 6.2 and Example 6.1.

Essential Question 14: See Table 6.3 and discussion in Section 6.

Essential Question 15: See discussion in Section 6 on page 35.

Essential Question 16: Answers will vary. Students may treat the equal sign operationally and give an answer of 11. Students may use different values for \square on each side of the equal sign to make a true number sentence (e.g., using 6 for \square on the left-hand side and 1 for \square on the right-hand side.) Students may ask whether the \square represents the same value on both sides of the equal sign.

Essential Question 17: See Exercise 5.13.

Section 5

Exercise 5.1: See Table 5.1.

Exercise 5.2: Assuming that person 1 knows person 2 and person 2 knows person 3 does not imply that person 1 knows person 3 (e.g., It is possible that Deb knows Judi and Judi knows Diane with Deb not knowing Diane).

Exercise 5.3: This relation is not an equivalence relation since the symmetric property fails (e.g., 2 is less than or equal to 3 but 3 is not less than or equal to 2).

Exercise 5.4: The ring properties can be thought of as the minimum requirements for a set with two operations to resemble the integers. See Section 7 for models that can be used to help students understand the various properties.

Exercise 5.5: The set of odd integers, $\{\dots, -5, -3, -1, 1, 3, 5, \dots\}$ does not contain the integer 0 since 0 is even. While the integer 0 is not in this set, we should make certain that there is no element x in the set such that $a + x = a$ and $x + a = a$ for all a in the set. (See Example 5.7.) This is clear since if such an element existed and was positive then adding it to any positive element in the set would always result in a value that is larger than either of the addends; and if such an element existed and was negative, then adding it to any negative element in the set would give a value that is more negative than either addend. [Note: If S is a subset of a ring R and S is also a ring, then S is called a subring of R . It can be shown that the additive identity element of a subring is the same as the additive identity element of the ring.]

Exercise 5.6: The set of even integers, $\{\dots, -6, -4, -2, 0, 2, 4, 6, \dots\}$ does form a ring. To show this we need to verify that each of the ring properties holds. We already know that the Associative for Addition, Commutative for Addition, and Distributive properties hold since they each hold for the set of integers. The closure property for addition and multiplication hold since the sum of any two even integers is an even integer and the product of any two even integers is an even integer. (You may want to verify this.) Also, 0 is an even integer (Why?), so the set of even integers has an additive identity element. Finally, additive inverses exist since the solution to the equation $a + x = 0$ for any a is $-a$ which is also even when a is even.

Exercise 5.7: The additive inverse for any element a is the integer x with the property $a \Delta x = 1$. (Remember, 1 is the additive identity element). Therefore, $x = 2 - a$ is the additive inverse element of a since $a \Delta (2 - a) = a + 2 - a - 1 = 1$.

Exercise 5.8: We have already shown that the additive identity element is 1 (see Example 5.7) and that the additive inverse of an element a is $2 - a$ (see Exercise 5.7). The rest of the properties can be shown to hold by applying the definitions of $a \Delta b$ and $a \blacktriangle b$ and using the fact that these properties hold for integers. (For example, to show that the Commutative Property of Addition holds, we need to show that $a \Delta b = b \Delta a$. This is true since $a \Delta b = a + b - 1 = b + a - 1$ since the addition on the right is ordinary addition of integers and the Commutative Property of Addition holds.) You should verify the rest of the properties.

Exercise 5.9: The first step in verifying that the set of rational numbers is a field is to define addition and multiplication and also remember that two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ are equal if and only if $ad = bc$.

$$\text{Addition: } \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \qquad \text{Multiplication: } \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

Also, recall that a , b , c , and d are integers. We now need to show that each of the properties in Table 5.4 holds. The closure properties hold since the addition and multiplication on the right-hand side of each definition is ordinary addition and multiplication of integers which is closed. Associative, Commutative, and the Distributive properties hold by manipulating the right-hand sides using the fact that these laws hold for integers. For example, the following shows that the Associative Property of Addition holds. Take note

of when manipulations are due to the definition of addition on the rational numbers and when the manipulations are due to addition and multiplication on the integers. In fact, please take the time to fill in the details.

$$\frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f} \right) = \frac{a}{b} + \left(\frac{cf + de}{df} \right) = \frac{a(df) + b(cf + de)}{b(df)} = \frac{(ad + bc)(f) + (bd)e}{(bd)f} = \frac{ad + bc}{bd} + \frac{e}{f} = \left(\frac{a}{b} + \frac{c}{d} \right) + \frac{e}{f}$$

The additive identity element is the integer 0 since for any rational number $\frac{a}{b}$ we have

$$\frac{a}{b} + 0 = \frac{a}{b} + \frac{0}{1} = \frac{a \cdot 1 + b \cdot 0}{b \cdot 1} = \frac{a}{b}. \text{ (Notice, again, when we use properties that hold for integers. For example,}$$

$a \cdot 1 = a$.) $0 + \frac{a}{b}$ is similar. Showing that $-\frac{a}{b}$ is the additive inverse of $\frac{a}{b}$ is also similar. The integer 1 can

be shown to be the multiplicative identity. Finally, the multiplicative inverse of an element $\frac{a}{b}$ is $\frac{b}{a}$ since

$$\frac{a}{b} \times \frac{b}{a} = \frac{ab}{ba} = \frac{ab}{ab} = ab \times (ab)^{-1} = 1. \text{ (The other way around is similar.)}$$

Exercise 5.10: We need to find an element m such that $a \blacktriangle m = a$ and $m \blacktriangle a = a$. The missing element that we are looking for (i.e., the multiplicative identity) is the integer 0 since $a \blacktriangle 0 = a + 0 - a(0) = a + 0 - 0 = a$. (Notice that we used the fact that 0 is the additive identity element for the integers.) It can also be shown that $0 \blacktriangle a = a$. Recall, the additive identity element was the integer 1.

Exercise 5.11:

$$\begin{aligned} a + b - b &= a + b + (-b) && \text{Definition of Subtraction} \\ &= a + (b + (-b)) && \text{Associativity for Addition} \\ &= a + 0 && \text{Additive Inverse Property} \\ &= a && \text{Additive Identity Property} \end{aligned}$$

; Students might explain that you are starting with some number and then adding another number to it, but then subtracting that number. So, you should end up with what you started with. They might show this using manipulatives.

Exercise 5.12: Adding $-c$ to both sides and using Associative, additive inverses, and the additive identity property we get

$$\begin{aligned} a + c + (-c) &= b + c + (-c) \\ a + (c + (-c)) &= b + (c + (-c)) \\ a + 0 &= b + 0 \\ a &= b \end{aligned}$$

Exercise 5.13:

$$\begin{aligned} a \times b \div b &= a \times b \times b^{-1} && \text{Definition of Division} \\ &= a \times (b \times b^{-1}) && \text{Associativity for Multiplication} \\ &= a \times 1 && \text{Multiplicative Inverse Property} \\ &= a && \text{Multiplicative Identity Property} \end{aligned}$$

; Students' justifications will vary but may include reasoning that states if you multiply a number by another number and then divide by this same number you end up with the number that you started with.

Exercise 5.14: Let a be an element of the field and suppose that b and c are both multiplicative inverses for a . Then, $ab = 1$ and $ac = 1$. And, $b = 1b = (ac)b = (ca)b = c(ab) = c1 = c$.

Exercise 5.15: Yes, since

$$\begin{aligned} a \times (b - c) &= a \times (b + -c) && \text{Definition of Subtraction} \\ &= (a \times b) + (a \times -c) && \text{Distributive Property} \\ &= (a \times b) + (-a \times c) && \text{See Table 5.7} \\ &= (a \times b) - (a \times c) && \text{Definition of Subtraction.} \end{aligned}$$

Alternatively (using commutativity), you might see that if you are taking c groups of a away from b groups of a that is the same as having $b - c$ groups of a (e.g., taking 2 groups of 8 away from 6 groups of 8 is the same as having $6 - 2$ or 4 groups of 8).

Exercise 5.16: Commutativity does not hold over subtraction. As an example, $6 - 2 \neq 2 - 6$; Commutativity does not hold over division. As an example, $8 \div 4 \neq 4 \div 8$; Associativity does not hold over subtraction. As an example, $15 - (6 - 2) \neq (15 - 6) - 2$; Associativity does not hold over division. As an example, $(24 \div 4) \div 2 \neq 24 \div (4 \div 2)$.

Exercise 5.17: The statement is not true as illustrated with the following counterexample.

$$3 \times (4 \times 2) \neq (3 \times 4) + (3 \times 2)$$

Section 6

Exercise 6.1: The following are the answers to this exercise. Explanations will vary. It is important that you consider your method of solution (i.e., did you use calculations or relational thinking – see the section for details): 11; 23; 6; 28, 40; T; T; T; T; T; F.

Exercise 6.2: See the discussion in Section 6.

Exercise 6.3: A key question was when Emma’s teacher asked Emma to find the number that belongs in the box below.

$$15 + 16 = 15 + \uparrow$$

Exercise 6.4: Number sentences will vary.

Exercise 6.5: Answers will vary.

Exercise 6.6: Part a) Mike’s response indicates that he is treating the equal sign as an operator or to signify to write the “answer”; part b) Strategies will vary, but may include masking techniques (e.g., covering up the circle on the right-hand side to allow students to focus on the first equality and subsequently masking the left-hand side); part c) Strategies will vary. See the discussion in Section 6 for strategies to help students build relational thinking. Strategies may include helping Alicia see why 8 groups of 14 objects (e.g., paperclips) is the same as 16 groups of 7 objects; part d) Anthony’s check marks seem to indicate that he is continuing the calculation and writing down the “answer” at each stage. That is, Anthony is treating the equal sign in an operational way; part e) To see that the number sentence is false, we just need to note that $3 + 7$ is not equal to 17.

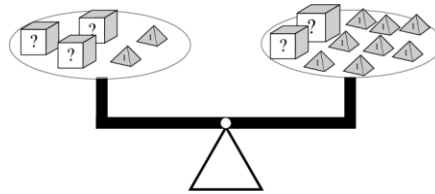
Exercise 6.7: The shapes are being used as unknowns since we are looking for specific values that make the equality statements true.

Exercise 6.8: Answers will vary. Sample answer: In the first equation, students might reason that 6 times 3 is 18 and 6 times 2 is 12; In the second equation, students might reason that $3 + 3 + 3 + 3 = 12$ and that $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2$ so that each box is $3\frac{1}{2}$.

Exercise 6.9: Answers will vary.

Exercise 6.10: Sample answer: Each sand bag must weigh 14 lb since $8 \times 3 = 24$, $6 \times 3 = 18$, and $8 + 6 = 14$.

Exercise 6.11: Sample Balance scale:



Each cube can be seen to weigh six pounds by removing two cubes from each side and two pyramids from each side.

Exercise 6.12: Answers will vary.

Exercise 6.13: Students might believe this number sentence can not be true since the symbols on both sides are not the same. The number sentence is true whenever $x = z$ (e.g., $3 + 8 = 8 + 3$).

Section 7

Exercise 7.1: Answers will vary. Table 7.1 provides some examples.

Exercise 7.2: Answers will vary.

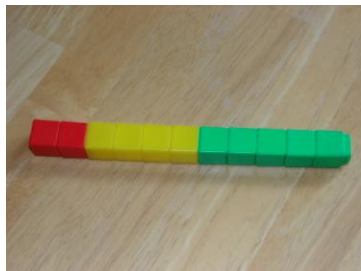
Exercise 7.3: Explanations will vary. Sample explanations (considering only positive numbers): As you divide a given number by numbers that are closer and closer to 0 the quotient gets larger and larger.

Recalling that division is defined in terms of multiplicative inverses, we could think of $\frac{1}{0} = 1 \times 0^{-1}$.

However, the inverse of 0 does not exist since it would be the number that you multiply by 0 to get 1 and any number times 0 is 0 (e.g., $8/0$ means to find a number a such that $0 \times a = 8$). Other explanations can be found at the Math Forum by exploring <http://mathforum.org/dr.math/faq/faq.divideby0.html>.

Exercise 7.4: Answers will vary. Students might use unifix cubes to help understand the Commutative Property of Addition. They might use two different colors and note that they have the same number of total cubes no matter what order they attach the colors (e.g., three green cubes followed by 4 red cubes is the same as 4 red cubes followed by 3 green cubes). Students might conjecture that when adding two numbers it doesn't matter what order you add those numbers in.

Exercise 7.5: Answers will vary. The following unifix cubes can be used to model $(2 + 4) + 5 = 2 + (4 + 5)$. Note that if you start by joining the red and yellow cubes and then join the green cubes the result (total number of cubes) is the same as joining the yellow and green cubes first and then adding the red cubes to this train of cubes.



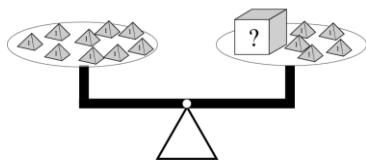
Section 8

Exercise 8.1: part a) Kris' response seems to indicate that he is viewing the equal sign as an operator and not a relation; however, Kris may understand alternate formats of equality; part b) First, calculate the sum of 7 and 2 to get 9 and then determine what number you add to itself twice to get 9; part c) One way would be to realize that $7 + 2$ is the same as $6 + 3$ and then see that each triangle must be $2 + 1$ or 3; part d) Students who haven't seen alternate forms of equality may think that the 'answer' (i.e., what number goes in the triangles) can not come before the equal sign; part e) Answers will vary; part f) Alisa seems to have an initial understanding of Benchmarks 1, 2, and 3.

Exercise 8.2: part a) Answers will vary, but note, that option C can be obtained by adding 14 and 14 and option D can be obtained by adding 14, 14, and 12; part b) Answers will vary, but note, students who choose options C or D may be treating the equal sign as an operator; part c) 55% (based on options C and D); part d) Sample answer: 12 is two less than 14, so the missing number must be 2 more than 14 or 16; part e) Answers will vary. A sample balance scale may contain 28 one-pound objects on the left-hand side and 12 one-pound objects on the right-hand side. Organizing the objects on the left-hand side into a group of twelve and a group of 14 reveals the answer.

Exercise 8.3: part a) Karen most likely calculated 3×18 ; part b) Answers will vary; part c) Sample number sentence: $\square + \square + \square = 18$

Exercise 8.4: part a) Ian – 2 pt, Lindsay – 0 pt, Mark – 1 pt; part b) Answers will vary. Sample balance scale:



part c) Since 8 is 2 more than 6 and 4 is two more than 2, we can use relational thinking to determine that the true number sentences must have 8 and 2 on one side of the equal sign and 6 and 4 on the other (i.e., $4 + 6 = 2 + 8$, $4 + 6 = 8 + 2$, $6 + 4 = 2 + 8$, $6 + 4 = 8 + 2$, $2 + 8 = 4 + 6$, $2 + 8 = 6 + 4$, $8 + 2 = 4 + 6$, $8 + 2 = 6 + 4$).

Exercise 8.5: Answers should include the position of the unknowns, the operations used, and the relationships among the numbers (e.g., in 15, since $4 \times 2 = 8$, the missing number must be half of 2 or 1).

Exercise 8.6: Ava – 0 pt, Ron – 3 pt (part b the answer is incorrect but shows the correct strategy), Quinn – 4 pt; part b) One way would be to reverse what is on each side of the equal sign (i.e., change the second number sentence to star equals heart minus 2); part c) Answers will vary, but may include giving Ava problems similar to Essential Question 15.

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Appendix A – Implied Cognitive Demand and Depth of Knowledge

A fundamental criterion used to develop the NECAP GLEs and GSEs is that the expectations should explicitly indicate cognitive demand (how content interacts with process) and that there should be a mix of cognitive demand levels at all grades. That is, one should not assume that students at lower grades do less cognitively demanding work. The cognitive demand or depth of knowledge required by an expectation or an assessment item is related to the number and strength of connections of concepts and procedures that a student needs to make to produce a response, including the level of reasoning required along with self-monitoring. Furthermore, there are additional factors that influence cognitive demand including contextual requirements, language, the number and variety of representations, requirements for generalizations to new situations, and the opportunity to learn.

It is important to note that depth of knowledge is not synonymous with difficulty. As an example, solving a multi-step linear equation with variables on both sides may be a difficult task for middle school students; however, the task can be solved by applying a standard procedure making the task of low complexity.

The NECAP states believe that expectations and assessment should be aligned in terms of their cognitive complexity. That is, the cognitive complexities of the assessment items should match that of the standards (what students are expected to know and be able to do). To ensure this alignment, the NECAP states have adopted Norman L. Webb’s (senior researcher with the Wisconsin Center for Educational Research) Depth of Knowledge classification system. Norman Webb’s system is based on four levels of classification. The full descriptions of each level are given on pages 4 and 5. The levels can be summarized as follows.

Level 1	Recall
Level 2	Skill/Concept
Level 3	Strategic Thinking
Level 4	Extended Thinking

The NECAP states, together with a committee of educators, analyzed the GLEs and GSEs for their implied cognitive demand. That is, all aspects of each expectation were analyzed and the implied cognitive demand levels were recorded. One of the charges of the NECAP test item review committees is to ensure that assessment items align not only with the expectations but also with their implied cognitive demands. The range of cognitive demands for each GLE and GSE is summarized in Table 1 on page 2. It should be noted that the highest level listed for each GLE and GSE should be thought of as a “ceiling” not a “target”. That is, the goal is to write items which cover the range of the levels indicated and not just the highest level. If one assesses only at the “target” level, all GLEs with a level 3 (for example) as their highest cognitive demand would only be assessed at level 3. This would potentially have two negative impacts on the assessment: 1) The assessment as a whole would be too difficult; and 2) important information about student learning along the achievement continuum would be lost. To the extent possible,

GLEs and GSEs should be assessed at the “ceiling” and at least one level below the “ceiling” in order to provide additional diagnostic information to educators. Furthermore, Table 2 shows an example of an expectation and how the different aspects of the expectation interact with Table 1.

Table 1

	Depth of Knowledge Levels for NECAP Assessment						
	2	3	4	5	6	7	10
M(N&O)–X–1	1, 2	1, 2	1, 2	1, 2	1, 2	1, 2	
M(N&O)–X–2	1	2	2	2	2	2	1, 2, 3
M(N&O)–X–3	1, 2	2	2	2,3	2,3		
M(N&O)–X–4		1, 2, 3	1, 2, 3	1, 2, 3	1, 2, 3	1, 2, 3	1, 2, 3
M(N&O)–X–5	1, 2						
M(G&M)–X–1	1, 2, 3	1, 2	1, 2	1, 2	1, 2		
M(G&M)–X–2						1, 2	1, 2, 3
M(G&M)–X–3			1, 2	1, 2	1, 2		
M(G&M)–X–4				1, 2		1, 2	2, 3
M(G&M)–X–5			1, 2		1, 2	1, 2, 3	1, 2, 3
M(G&M)–X–6	1, 2	1, 2	1, 2	1, 2	1, 2, 3	1, 2, 3	1, 2, 3
M(G&M)–X–7	This GLE will NOT be directly assessed but embedded in problems in other content strands.						1, 2
M(G&M)–X–8							
M(G&M)–X–9							2, 3
M(F&A)–X–1	2	2	2	2	2, 3	2, 3	2, 3
M(F&A)–X–2					1, 2	1, 2, 3	1, 2, 3
M(F&A)–X–3			1	1	1, 2	1, 2	1, 2
M(F&A)–X–4	1	1, 2	1, 2	1, 2	1, 2	1, 2	
M(DSP)–X–1	1, 2, 3	1, 2, 3	1, 2, 3	1, 2, 3	1, 2, 3	1, 2, 3	2, 3
M(DSP)–X–2	2, 3	2, 3	2, 3	2, 3	2, 3	2, 3	2, 3
M(DSP)–X–3		1, 2		1, 2		2, 3	1, 2, 3
M(DSP)–X–4	2		2, 3		2, 3		1, 2, 3
M(DSP)–X–5		1, 2	1, 2	1, 2	1, 2, 3	1, 2, 3	1, 2, 3

Black cells indicate GLEs or GSEs that are not assessed on NECAP at the given level.

Sample Mathematics GLE* for End of Grade 6	Potential DoK Levels	DoK Ceiling	Aspects of GLE at different levels**
<p>M(F&A)–6–1 Identifies and extends to specific cases a variety of patterns (linear and nonlinear) represented in models, tables, sequences, <u>graphs</u>, or in problem situations; or writes a rule in words or symbols for finding specific cases of a linear relationship; or <u>writes a rule in words or^{sc} symbols for finding specific cases of a nonlinear relationship</u>; and <u>writes an expression or^{sc} equation using words or^{sc} symbols to express the generalization of a linear relationship (e.g., twice the term number plus 1 or^{sc} $2n + 1$).</u></p>	2, 3	3	<p>Level 2 Extends a pattern to a specific case Level 3 Generalizes a pattern</p>

Table 2

*GLE NOTES: Underlining in the GLE indicates that this concept or skill is “new” to grade 6 for assessment purposes. The superscript “sc” indicates that students have a choice in how they complete the task (e.g., students can use words **or** symbols to express the rule).

**Recall, one must also consider other factors when making decisions on Depth of Knowledge levels such as contextual requirements, language, the number and variety of representations, requirements for generalizations to new situations, and the opportunity to learn.

Depth of Knowledge Descriptors for Mathematics
Norman L. Webb
March 28, 2002

Mathematics Depth of Knowledge Levels

Level 1 (Recall) includes the recall of information such as a fact, definition, term, or a simple procedure, as well as performing a simple algorithm or applying a formula. That is, in mathematics a one-step, well-defined, and straight algorithmic procedure should be included at this lowest level. Other key words that signify a Level 1 include “identify,” “recall,” “recognize,” “use,” and “measure.” Verbs such as “describe” and “explain” could be classified at different levels depending on what is to be described and explained.

Level 2 (Skill/Concept) includes the engagement of some mental processing beyond a habitual response. A Level 2 assessment item requires students to make some decisions as to how to approach the problem or activity, whereas Level 1 requires students to demonstrate a rote response, perform a well-known algorithm, follow a set procedure (like a recipe), or perform a clearly defined series of steps. Keywords that generally distinguish a Level 2 item include “classify,” “organize,” “estimate,” “make observations,” “collect and display data,” and “compare data.” These actions imply more than one step. For example, to compare data requires first identifying characteristics of the objects or phenomenon and then grouping or ordering the objects. Some action verbs, such as “explain,” “describe,” or “interpret” could be classified at different levels depending on the object of the action. For example, if an item required students to explain how light affects mass by indicating there is a relationship between light and heat, this is considered a Level 2. Interpreting information from a simple graph, requiring reading information from the graph, also is a Level 2. Interpreting information from a complex graph that requires some decisions on what features of the graph need to be considered and how information from the graph can be aggregated is a Level 3. Caution is warranted in interpreting Level 2 as only skills because some reviewers will interpret skills very narrowly, as primarily numerical skills, and such interpretation excludes from this level other skills such as visualization skills and probability skills, which may be more complex simply because they are less common. Other Level 2 activities include explaining the purpose and use of experimental procedures; carrying out experimental procedures; making observations and collecting data; classifying, organizing, and comparing data; and organizing and displaying data in tables, graphs, and charts.

Mathematics Depth of Knowledge Levels continued

Level 3 (Strategic Thinking) requires reasoning, planning, using evidence, and a higher level of thinking than the previous two levels. In most instances, requiring students to explain their thinking is a Level 3. Activities that require students to make conjectures are also at this level. The cognitive demands at Level 3 are complex and abstract. The complexity does not result from the fact that there are multiple answers, a possibility for both Levels 1 and 2, but because the task requires more demanding reasoning. An activity, however, that has more than one possible answer and requires students to justify the response they give would most likely be a Level 3. Other Level 3 activities include drawing conclusions from observations; citing evidence and developing a logical argument for concepts; explaining phenomena in terms of concepts; and using concepts to solve problems.

Level 4 (Extended Thinking) requires complex reasoning, planning, developing, and thinking most likely over an extended period of time. The extended time period is not a distinguishing factor if the required work is only repetitive and does not require applying significant conceptual understanding and higher-order thinking. For example, if a student has to take the water temperature from a river each day for a month and then construct a graph, this would be classified as a Level 2. However, if the student is to conduct a river study that requires taking into consideration a number of variables, this would be a Level 4. At Level 4, the cognitive demands of the task should be high and the work should be very complex. Students should be required to make several connections—relate ideas *within* the content area or *among* content areas—and have to select one approach among many alternatives on how the situation should be solved, in order to be at this highest level. Level 4 activities include designing and conducting experiments; making connections between a finding and related concepts and phenomena; combining and synthesizing ideas into new concepts; and critiquing experimental designs.

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