Counting Techniques and Probability

A Research Based Unit of Study for Middle School Teachers

Rhode Island Department of Education
Office of Instruction
December 2007

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Section 1: Purpose and Design

Probability and Statistics, along with counting techniques, are important topics in middle school mathematics that have gained emphasis in the curriculum and are intended to span across grades. *Counting Techniques and Probability* is one unit of study in a series of units designed around the big ideas in middle school mathematics.

The purpose of this unit of study is to support quality instruction by increasing teachers' mathematical content knowledge using research findings related to students' understandings of probability, and examining teaching recommendations based upon those findings. When adapting the ideas in these units of study to the classroom, teachers should purposefully design their lesson plans to incorporate three important principles of learning as identified by the National Research Council in *How Students Learn – Mathematics in the Classroom* and described in Table 1.1 on the following page (Donovan & Bransford, 2005). This unit is not meant to supplant current curricula materials, but rather to be used in conjunction with them. The three principles described in Table 1.1 are modeled throughout this unit of study. The Essential Questions in Section 3 seek to engage preconceptions, and the various sections throughout the unit connect procedural knowledge to conceptual knowledge. There are checkpoints along the way to monitor and reflect on your progress. Additionally, the summary section (Section 8) contains exercises allowing the participants to reflect upon their instructional programs and curricular materials.

Each unit of study begins by examining the Grade-Level Expectations that are pertinent to the particular unit of study. Following the identification of the expectations related to the unit of study, you will answer some essential questions. It is recommended that you answer these essential questions individually prior to reading subsequent sections. The essential questions will help frame the mathematical ideas of the unit. Exercises appear throughout the remaining sections. These exercises are imbedded within the section rather than at the end of the section and are intended to be solved and discussed as you are working through the section. Section 7 is devoted to examining NECAP released items and the student work that is available for these items. This section is subsequent to the sections that discuss research findings. So, once you reach this section, you will be able to make connections between the research and the NECAP items and identify typical student misconceptions.

This unit of study does not attempt to cover all areas of probability, nor does it attempt to be a guide of activities. The focus of the unit is intentionally on the research behind the aspects of this topic that middle school students study. While some content is reviewed in Section 4, a more in-depth content treatment can be found by examining the references in Section 10. Furthermore, Section 9 contains links to rich standards-based resources and activities dealing with the concept of probability.

Additionally, students need opportunities to work collaboratively, share ideas, and present ideas. Their understandings should be challenged and students should be allowed to build and construct knowledge for themselves. Students should actively engage in mathematics. Teachers should carefully guide this work, and should be cautious about just ‘telling students the answers.’ Teachers should ask students to explain how they know and allow them to share multiple ways to solve problems. Teachers need to continually probe students’ understandings, especially by allowing students to explore new ideas on their own. Teachers need to resist modeling a few dozen low-level problems (i.e., not cognitively challenging) for students and then asking them to solve far too many homework problems all of which can be matched to one of the modeled problems.

Table 1.1 – Principles of Learning

Principle	Description
Principle 1 – Engaging Preconceptions	Students come to the mathematics classroom with ideas about the structures of mathematics and informal understandings. If their preconceptions are not engaged and if there is no bridge between informal and formal understanding, students may have difficulty learning new ideas and may continue to revert to their preconceived notions.
Principle 2 – Connecting Procedural/factual knowledge and Conceptual Understanding	Procedural knowledge and conceptual understanding must be balanced. When one places too much emphasis on procedural fluency, the result is a lack of understanding in how the procedures work. Whereas, when one places too much emphasis on conceptual knowledge, often students lack the ability to perform the procedures in an efficient way. Teachers must help students build and connect ideas and organize knowledge into networks. It is important to discuss various solution methods and why they work and make connections among them.
Principle 3 – Self Monitoring	Students need to be afforded the opportunity to think about their own learning and assess their own mathematical progress. Eventually, such assessment opportunities will be internalized and students will begin to monitor their own progress.

Exercise 1.1 Take some time to complete the following graphic organizer in your journal. You should use this organizer throughout the unit of study and continually add to it. If you are working with colleagues, it is recommended that you each complete your own graphic organizer and post it on chart paper so that you can learn throughout the unit from the work of your colleagues. It is okay if there are portions of the graphic organizer that you can not initially fill out. Again, you should continually return to this graphic organizer as you complete the various sections.

Probability is…	**Things that I'm unclear about regarding counting techniques and probability/What I want to learn…**
Students' misconceptions with counting techniques and probability include…	**Teaching strategies for dealing with students' difficulties with probability concepts include…**
Probability and Statistics are connected because…	**Things I learned include…**

Section 2: Connecting to the Grade-Level Expectations

One of the goals of this unit of study is to help you become more familiar with the Grade-Level Expectations dealing with counting techniques and probability. In order to achieve this goal, you will spend some time looking through the Grade-Level Expectations (see Exercise 2.1) to find the standards related to counting techniques and probability.

In Section 7, you will spend time analyzing released NECAP items and student work associated with these items. Please complete Exercise 2.1 before reading on.

Exercise 2.1 Locate the GLEs that deal with counting techniques and probability. Make certain that you are considering both state and local GLEs.

> Please refer back to the *About the Mathematics GLE* page at the beginning of the GLEs if necessary to recall how bolding and underlining is used in the document, along with "or^{sc}". On the following page, assessment GLEs appear in bold boxes; all others are local GLEs.

Exercise 2.2 Locate the Curriculum Focal Points and Connections at middle grades that deal with counting techniques and probability.

This unit of study will primarily deal with content contained within the following GLEs (New Hampshire Department of Education & Rhode Island Department of Education, 2005).

Grade 6

M:DSP:6:4 **Uses counting techniques to solve problems** in context involving combinations or simple permutations using a variety of strategies (e.g., organized lists, tables, tree diagrams, models, Fundamental Counting Principle, or[sc] others).

M:DSP:6:5 **For a probability event in which the sample space may or may not contain equally likely outcomes, predicts** the theoretical probability of an event and tests the prediction through experiments and simulations; and designs fair games.

M:DSP:6:5 **For a probability event in which the sample space may or may not contain equally likely outcomes, determines** the experimental or theoretical probability of an event in a problem-solving situation.

M:DSP:6:6 **In response to a teacher or student generated question or hypothesis** decides the most effective method (e.g., survey, observation, experimentation) to collect the data (numerical or categorical) necessary to answer the question; collects, organizes, and appropriately displays the data; analyzes the data to draw conclusions about the question or hypothesis being tested, and when appropriate makes predictions; and asks new questions and makes connections to real world situations.

(IMPORTANT: *Analyzes data consistent with concepts and skills in M:DSP:6:2.*)

Grade 7

M:DSP:7:4 **Uses counting techniques to solve problems** in context involving combinations or permutations (e.g., How many different ways can eight students place first, second, and third in a race?) using a variety of strategies (e.g., organized lists, tables, tree diagrams, models, Fundamental Counting Principle, or[sc] others).

M:DSP:7:5 **For a probability event in which the sample space may or may not contain equally likely outcomes, predicts** the theoretical probability of an event and tests the prediction through experiments and simulations; and compares and contrasts theoretical and experimental probabilities.

M:DSP:7:5 **For a probability event in which the sample space may or may not contain equally likely outcomes, determines** the experimental or theoretical probability of an event in a problem-solving situation.

M:DSP:7:6 **In response to a teacher or student generated question or hypothesis** decides the most effective method (e.g., survey, observation, experimentation) to collect the data (numerical or categorical) necessary to answer the question; collects, organizes, and appropriately displays the data; analyzes the data to draw conclusions about the question or hypothesis being tested while considering the limitations that could affect interpretations; and when appropriate makes predictions; and asks new questions and makes connections to real world situations.

(IMPORTANT: *Analyzes data consistent with concepts and skills in M:DSP:7:2*.)

Grade 8

M:DSP:8:4 **Uses counting techniques to solve problems** in context involving combinations or permutations using a variety of strategies (e.g., organized lists, tables, tree diagrams, models, Fundamental Counting Principle, or[sc] others).

M:DSP:8:5 **For a probability event in which the sample space may or may not contain equally likely outcomes, determines** the experimental or theoretical probability of an event in a problem-solving situation; and **predicts** the theoretical probability of an event and tests the prediction through experiments and simulations; and compares and contrasts theoretical and experimental probabilities.

M:DSP:8:6 **In response to a teacher or student generated question or hypothesis** decides the most effective method (e.g., survey, observation, experimentation) to collect the data (numerical or categorical) necessary to answer the question; collects, organizes, and appropriately displays the data; analyzes the data to draw conclusions about the question or hypothesis being tested while considering the limitations that could affect interpretations; and when appropriate makes predictions; and asks new questions and makes connections to real world situations.

(IMPORTANT: *Analyzes data consistent with concepts and skills in M:DSP:8:2*.)

Section 3: Essential Questions

Essential questions help you to begin to think about the mathematics that will be the focus of this unit of study. You are encouraged to think deeply about these questions and to work them independently before discussing your thoughts with colleagues and before reading subsequent sections. It is also recommended that you keep a journal that contains your work on these problems and the problems throughout this unit of study. You are encouraged to use a PEN in your journal so that you can not easily erase your work. Even though this is contrary to what many mathematics teachers ask students to do, using pen will allow you to go back and reflect on your initial thoughts, analyze any errors that you have made, and see how your learning has developed. (This is a suggestion that Tim Kurtz, NH State Assessment Director, has passed on to me that I try to share whenever possible. As a teacher, if you require your students to use pen, you will be able to easily identify what students were thinking when working problems and any errors made by students. This will facilitate your efforts in addressing students' preconceptions and misconceptions and will allow students to monitor their progress – See Table 1.1.) Some of these questions are intentionally vague in some areas; the reasons will be apparent when one works through the remainder of the sections. Many questions will be discussed throughout various sections of this unit of study; therefore, please refrain from looking at the answers to the essential questions (Section 9) until working through the entire unit of study. Additional questions will be posed throughout the various sections.

Essential Question 1 A fair coin is tossed six times and the results are recorded in the order they appear. At each toss, the coin lands either H (heads) or T (tails).

The outcome H T H T T H

☐ is less likely than
☐ is as likely as
☐ is more likely than

the outcome H H H H H H.

A fair coin is tossed six times [and lands either H (heads) or T (tails)]. An outcome that contains three heads and three tails

☐ is less likely than
☐ is as likely as
☐ is more likely than

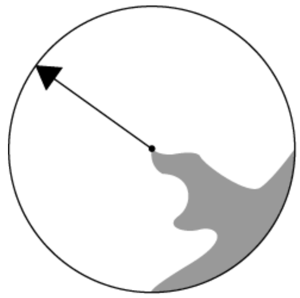
an outcome that has six heads.

Explain your reasoning.

(Adapted from Callaert, 2002)

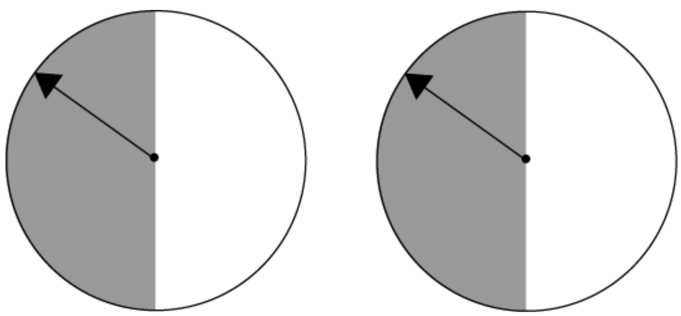
Essential Question 2 The arrow on the spinner below is spun once. What is the approximate probability that the tip of the arrow will land in the shaded region? Explain your reasoning.

a) 0
b) 1/10
c) 1/4
d) 1/2
e) 1



(Adapted from Shaughnessy, Barrett, Billstein, Kranendonk, & Peck, 2004)

Essential Question 3 Alisa said that if each arrow on each spinner below is spun simultaneously there is a 50% chance that both arrows will land on the gray section. Explain whether or not you agree with Alisa.



(Adapted from Mitchell, Hawkins, Jakwerth, Stancavage, & Dossey, 1999)

Essential Question 4 Assume that the chance of having a baby boy or girl is the same. Over the course of a year, in which type of hospital would you expect there to be more days on which at least 60% of the babies born were boys?

A) In a large hospital
B) In a small hospital
C) It makes no difference

Explain your reasoning.

(Adapted from Shaughnessy & Bergman, 1993)

Essential Question 5 There are two cab companies that operate in the city, a Blue Cab company and a Green Cab company. It is known that 85% of the cabs in the city are Green and 15% are Blue. A cab was involved in a hit-and-run accident at night. A witness at the scene identified the cab involved in the accident as a Blue cab. This witness was tested under similar visibility conditions, and made correct color identifications in 80% of the trial instances. Given that the witness identified the cab as blue, what is the probability that the cab involved in the accident was a Blue cab rather than a Green one? Explain your reasoning.

(Adapted from Shaughnessy & Bergman, 1993)

Essential Question 6 All families of six children in a city were surveyed. In 72 families the exact order of births of boys and girls was G B G B B G. What is your estimate of the number of families surveyed in which the exact order of births was B G B B B B? Explain your reasoning. [Assume that the probability of boy and girl births were equal.]

(Adapted from Konold, 1995)

Essential Question 7 Which event is more likely – obtaining three heads by tossing a fair coin three times or obtaining three heads by tossing three fair coins simultaneously? Explain.

Essential Question 8 The Sutton Meteorological Center wanted to determine the accuracy of the weather forecasts. They searched their records for those days when the forecaster had reported a 70% chance of rain. They compared these forecasts to records of whether or not it actually rained on those particular days.

The forecast of 70% chance of rain can be considered very accurate if it rained on:

a) 95% - 100% of those days.
b) 85% - 94% of those days.
c) 75% - 84% of those days.
d) 65% - 74% of those days.
e) 55% - 64% of those days.

Explain your reasoning.

(Adapted from Konold, 1995)

Essential Question 9 For the following questions, assume that ‘birthday’ implies the same month and day of the month.

a) How many people do you need in a room to be certain (100% sure) that at least two people share the same birthday? Explain.

b) How many people do you need in a room to have about a 50% chance that at least two people share the same birthday? Explain.

c) How many people do you need in a room to have about a 50% chance that at least two people share a particular birthday (e.g., November 11)? Explain.

Essential Question 10 Suppose you were to spin a penny 40 times. How many times would you expect the penny to land on heads? Explain. How would you go about determining whether or not your prediction was accurate?

Essential Question 11 Suppose you asked each of your students to toss a fair coin 100 times and record the sequence of Heads (H’s) and Tails (T’s) generated. And, suppose that you told your students that some of them were allowed to ‘cheat.’ That is, some of them are allowed to just write down a random sequence of H’s and T’s that they think they might get if they were to actually perform the experiment. You instruct that some students must actually perform the experiment. Additionally, you tell your students that you will leave the room and they are able to decide who will ‘cheat’ and who will perform the experiment. Then, the students record this information and place their sequences in an envelope with either their name on the back or some other way of identifying the sequence so they know if it was generated by tossing the coin or by cheating. How might you determine which sequences were actually generated by tossing the coin and which by cheating?

Essential Question 12 How does probability connect to statistics? Explain.

The next section briefly covers some of the key ideas and results in probability which represent the underlying content of this unit of study. The section will also discuss some various difficulties that students exhibit with counting techniques.

Section 4: Counting Techniques and Results from Probability

While it can be quite complex to count the number of possible outcomes for a given experiment, some facility with counting techniques will lend understanding to many results in probability. Even though this unit of study assumes that you have some understanding of counting techniques and probability, and the major focus is on understanding students' conceptions and teaching strategies, we will establish some key definitions in this section and work through some examples and exercises to illustrate those definitions and present a variety of solution strategies. If you are looking for a more thorough treatment of the content, it is recommended that you consult your curricular resources or those in the reference section. In Section 6, we will learn that researchers have shown that students often have difficulty listing the sample space for an experiment, and realizing whether or not that sample space contains equally likely outcomes. Exercises like those presented in this section will allow students to engage in activities that challenge their understandings of the sample space. Students should be allowed to present their varied approaches to solving these problems and should discuss whether or not the sample spaces contain equally likely outcomes.

We begin this section by reviewing some key definitions and results, as shown in Table 4.1. Examples follow.

Table 4.1 – Key Definitions and Results

Definition/Result	Description
Sample Space	The set of all possible disjoint outcomes for a probability experiment
Event	A subset of the sample space
Theoretical Probability of an Event	The ratio of the number of outcomes in the event to the total number of possible disjoint equally likely outcomes
Empirical Probability of an Event (Experimental Probability)	The probability of an event occurring based upon an experiment – the number of times the event occurred in the experiment divided by the number of trials (as the number of trials increase, the empirical probability approaches the theoretical probability – See the Law of Large Numbers in Section 6)
Fundamental Counting Principle (Multiplication Principle)	If some choice can be made in M different ways and another choice can be made in N different ways, then these two choices can be made in $M \times N$ different ways (this idea can be extended to k choices)

Table 4.1 Continued

Combination	The number of ways of picking *k* outcomes from *n* possible outcomes when the ordering of the outcomes isn't important
Permutation	The number of ways of picking *k* outcomes from *n* possible outcomes when the ordering of the outcomes is important
Tree Diagram	A graphic organizer that resembles the branches of a tree that is used to count the number of possible outcomes of an experiment or to calculate the probability of an event
Area Model	A model used where the amount of area designated for each outcome corresponds to the probability of that outcome occurring

Example 4.1 – Examples of Definitions from Table 4.1

If we toss a six-sided number cube that contains the numbers 1-6 on the faces, where no number is repeated, then:

- the experiment is tossing the number cube;
- a possible sample space is {1, 2, 3, 4, 5, 6};
- another possible sample space is {multiple of 3, not a multiple of 3}, where we are restricted to whole numbers between 1 and 6, inclusive;
- we might consider the event E = {obtaining an even number on a single toss}.

Notice that both sample spaces contain disjoint outcomes that together cover all possible outcomes for the experiment. However, the second sample space does not contain equally likely outcomes. This can be seen by looking at the first sample space and seeing that a multiple of 3 can happen in two ways (i.e., obtaining a 3 or 6 on a single toss); whereas, not a multiple of 3 can happen in four ways (i.e., obtaining a 1, 2, 4, or 5 on a single toss). Also, note that the probability of event E occurring, denoted by $P(E)$, is $\frac{3}{6}=\frac{1}{2}$, since tossing an even number can occur in 3 out of 6 ways.

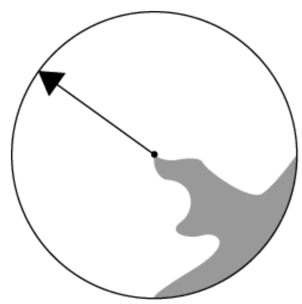
Example 4.2 – Some Notes about Sample Space

Suppose a bag contains 3 black marbles and 4 red marbles and you are to reach into the bag and pick one marble at random. One could express the sample space as {black, red}; however, it would not contain equally likely outcomes since obtaining a black marble can occur in 3 ways, and obtaining a red marble can occur in 4 ways. Thus, you would not say that the probability of obtaining a black marble is 1 out of 2 or $\frac{1}{2}$, since the definition requires us to have equally likely outcomes. You might consider labeling each of the marbles. For example, label the first red marble as R_1, the second red marble as R_2, and so on. This would create the following sample space consisting of seven equally likely outcomes: $\{B_1, B_2, B_3, R_1, R_2, R_3, R_4\}$. From here, it is straightforward to see that the probability of obtaining a black marble on a single random draw from the bag is 3 out of 7 or $\frac{3}{7}$. Sample space will be discussed in further detail in Section 5.

Exercise 4.1 Suppose you were to toss a six-sided number cube twice (that contains the numbers 1-6 on the faces where no number repeats) and record the sum of the numbers that land face up. How many equally likely outcomes does the sample space contain? How many distinct outcomes does the sample space contain?

Example 4.3 – Empirical Probability of an Event

Take a moment to revisit Essential Question 2 from Section 3.

> **Essential Question 2** The arrow on the spinner below is spun once. What is the approximate probability that the tip of the arrow will land in the shaded region? Explain your reasoning.
>
> a) 0
> b) 1/10
> c) 1/4
> d) 1/2
> e) 1
>
> 
>
> (Adapted from Shaughnessy et al., 2004)

Many students may initially believe that the probability that the arrow lands in the shaded-gray region is close to 1/10, since the area of the shaded-gray region is about 1/10 the area of the circle. Performing the experiment and having students fill in the table below can help them to see that the proportion of shaded outcomes in the short term may not be stable, but over time the relative frequency of an outcomes settles down to the true probability. Based on the experiment, students should find that their empirical probability is close to 1/4. Take a moment to justify why this would be true. Record your thoughts in your journal.

Spin Number	Landed in Shaded region (Y or N)	Cumulative Number of Spins in Shaded Region	Cumulative Number of Spins	Proportion of Spins in Shaded Area
1				
2				
3				
. . .				

Example 4.4 – Fundamental Counting Principle

Each of the following problems illustrates the use of the Fundamental Counting Principle:

- Crystal is choosing an outfit from 3 blouses, 4 pairs of pants, and 2 pairs of shoes. How many different outfits can she choose (an outfit consists of 1 blouse, 1 pair of pants, and 1 pair of shoes)?

- Suppose six students were lining up for lunch. In how many different orders could those six students line up?

- How many different telephone numbers are possible within the 603 area code?

- How many possible equally likely outcomes are there when rolling a six-sided number cube (with faces labeled 1-6 and no repeating numbers) and flipping a coin (with heads on one side and tails on the other side)?

- A lottery consists of choosing 6 numbers from 45 total numbers. How many possible equally likely outcomes are there?

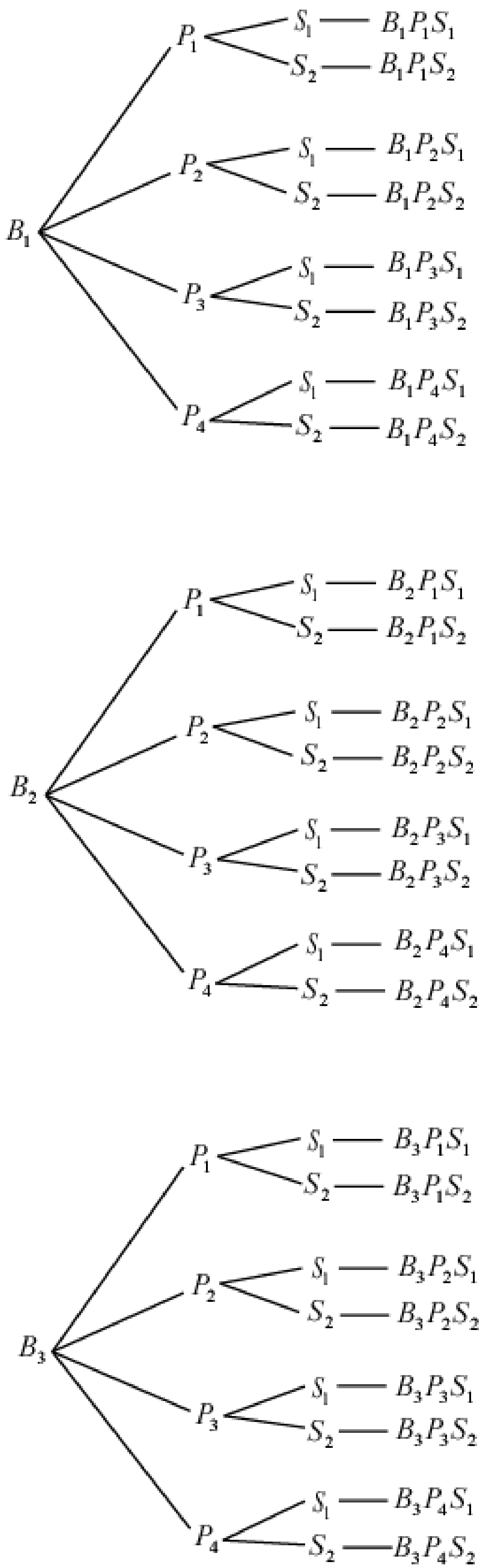
- A three person art council is being formed. The council will consist of 1 representative from a group of 5 artists, 1 representative from a group of 4 photographers, and 1 representative from a group of 6 musicians. How many different councils can be formed?

- From a group of 10 people, a 6 person committee is being formed. How many different committees are possible?

- How many different ways can 7 people be seated at a round table?

Notice that many strategies exist for solving these problems. For example, in the first task, the Fundamental Counting Principle can be directly applied to determine that there are $3 \times 4 \times 2 = 24$ different possible outfits. That is, for each of the 3 blouses there are 4 choices of pairs of pants, and for each of the 4 choices of pairs of pants there are 2 choices for shoes. Allowing students to use multiple strategies can help contribute to conceptual understanding. A tree diagram clearly shows the previous explanation (B_1 stands for blouse 1, and so on…).

Tree diagram showing number of possible outfits, where B_1 stands for blouse 1, and so on….



This section purposely avoids the typical combination and permutation formulas that are often found in textbooks. It is important that students develop a strong conceptual foundation for permutation and combination problems before they begin to apply the formulas. For example, consider trying to determine how many 3 person committees can be formed from a selection of 4 people. Directly applying the fundamental counting principle leads to $4 \times 3 \times 2 = 24$ different ways. However, in this case, you must realize that the order of the members within the committee isn't important. That is, if we select Ron, Whisper, and Ava (in that order) for the committee that is the same as selecting Whisper, Ava, and Ron (in that order). Thus, 24 is an over count of the total number of different committees by the number of ways that we can rearrange 3 objects – which would be $3 \times 2 \times 1 = 6$ ways. Therefore, there are 4 different committees possible. One approach would be to use a chart to indicate which three people we select from the four.

Ron	**Whisper**	**Ava**	**Adrian**
X	X	X	
X	X		X
X		X	X
	X	X	X

Exercise 4.2 Use counting techniques to solve each of the problems in bullets 2-8 from Example 4.4 (avoid just applying combination and permutation formulas and pay particular attention to whether or not order matters).

Exercise 4.3 Explain how to use the Fundamental Counting Principle to find the number of positive divisors of 2160.

Example 4.5 – Using Counting Techniques to Compute Probabilities

Take a moment to revisit Essential Question 9 from Section 3.

> **Essential Question 9**
>
> a) How many people do you need in a room to be certain (100% sure) that at least two people share the same birthday? Explain.
>
> b) How many people do you need in a room to have about a 50% chance that at least two people share the same birthday? Explain.
>
> c) How many people do you need in a room to have about a 50% chance that at least two people share a particular birthday (e.g., November 11)? Explain.

We will illustrate how counting techniques can be used to solve this problem. While the problem is a nice exercise in utilizing counting techniques, it is also a good example of how surprising results can occur – or, how results that are often counterintuitive arise in counting problems and in the study of probability.

A concept known as the Pigeonhole Principle helps to answer the original question. The Pigeonhole Principle is named after the series of compartments that you might find in a desk for sorting papers. Or, for example, the mail slots that you might find behind a hotel desk. The principle states that if there are a fixed number of slots, say *n*, and *k* items to go into those slots where $k > n$, then it must be the case that at least one slot contains more than one item (e.g., consider 14 letters that must go into 10 slots). While the statement of the principle is quite simple and very intuitive, it can be quite powerful when working with numeric problems as illustrated in Exercise 4.7. Invoking the Pigeonhole Principle, we see that the answer to the original question is 367 (counting February 29). Thus, we might assume that we would need about half of 365 or approximately 183 people in a room to be about 50% certain that at least two people share the same birthday. However, this initial guess is no where near the actual number. To see why, let's begin by assuming that we have two people in a room. Then, by the multiplication principle, there would be 365 × 365 pairs of dates (ignoring February 29) that could represent the birthdays of the two people in the room. And, 365 × 364 of those pairs are different (whatever the date of the first person's birthday, there are 364 choices for the second person's birthday to guarantee that they are not the same). Another way of thinking about this is to organize a table, where the dates down the left-hand side represent all possible days for the first person's birthday, and the dates across the top represent all possible days for the second person's birthday.

		Second Person			
		Jan. 1	**Jan. 2**	**Jan. 3**	**…**
First Person	**Jan. 1**	Jan. 1, Jan. 1	Jan. 1, Jan. 2	Jan. 1, Jan. 3	…
	Jan. 2	Jan. 2, Jan. 1	Jan. 2, Jan. 2	Jan. 2, Jan. 3	…
	Jan. 3	Jan. 3, Jan. 1	Jan. 3, Jan. 2	Jan. 3, Jan. 3	…
	⋮	⋮	⋮	⋮	⋮

Notice that the 365 pairs along the diagonal represent the pairs where the two people could share the same birthday.

So, the probability that these two people **don't** share the same birthday is $\frac{365 \times 364}{365 \times 365}$. To determine the probability that these two people do share the same birthday, we just need to subtract this result from 1 (since the probability of an event is a number between 0 and 1, inclusive; where 0 represents an impossible event and 1 represents a certain event). Then, the probability that these two people do share the same birthday is $1 - \frac{365 \times 364}{365 \times 365} \approx 0.003$ -- which isn't very likely.

Continuing this logic, if there were three people in a room, there would be 365 × 365 × 365 possible triples of dates, and 365 × 364 × 363 possible triples where none of the three dates are the same. So, the probability that none of the three people in the room share the same birthday is $\frac{365 \times 364 \times 363}{365 \times 365 \times 365}$, making the probability that at least two people share the same birthday $1 - \frac{365 \times 364 \times 363}{365 \times 365 \times 365} \approx 0.008.$ Extending to 23 people, we find that the probability of at least two people sharing the same birthday is about 0.507. Therefore, if there are 23 people in a room, there is about a 50% chance that at least two of those people share the same birthday. In fact, if there were about 40 people in a room, the probability of at least two people sharing the same birthday would be around 89% (take a moment to verify this).

Now, notice that the third question is quite different from the second question. In fact, there was a Johnny Carson episode where a guest was trying to explain the second question. Johnny Carson didn't believe the guess and asked how many people in the audience shared his birthday – March 19. The audience had about 120 guests and no one in the audience shared his birthday. The guest failed to realize that the second question is about having some birthday in common, not a particular birthday in common. (Paulos, 2001).

Let's suppose we wanted to find how many people would need to be in the audience to be about 50% certain that at least one person would share the same birthday with Johnny – March 19. Again, let's start simple. Let's suppose there were two people in the audience. Then, if we simply take out the row and column in the chart above labeled March 19, we will find all of the ways in which the two people do not have March 19 as a birthday. Notice that this is 364 × 364 ways. Extending this idea, if there were *n* people in the audience there would be 364^n ways that these people could not have March 19 as their birthday, making the probability that at least one of these *n* people was born on March 19 equal to $1 - \frac{364^n}{365^n}$. And, this expression is approximately 0.50 when $n = 253$. So, there would need to be about 253 people in the audience to have about a 50% chance that at least one person shares his or her birthday with Johnny. The point of this is that some unlikely event is likely to occur, whereas it is less likely that a particular one will occur (Paulos, 2001).

Exercise 4.4 Explore the lesson *Birthday Paradox* from the illuminations web site at http://illuminations.nctm.org/LessonDetail.aspx?id=L299. This activity uses the graphing calculator to run a simulation to examine the second part of Essential Question 9.

Exercise 4.5 Use the Pigeonhole Principle to prove that there are two non-bald people on the planet that have the same number of hairs on their bodies. (Adapted from Burger & Starbird, 2005)

When working with counting problems with students, one should be aware that students often have difficulty with the concepts illustrated in Table 4.2 (Batanero & Sanchez, 2003). While some of these concepts are typically introduced in high school (e.g., permutations with indistinguishable objects), it is important to understand that these difficulties arise. In fact, if students merely use combination and permutation formulas without understanding them (e.g., why they are dividing by 4! in a particular situation), they are likely to struggle when these formulas do not directly apply (e.g., finding permutations where some of the objects are not distinguishable). However, taking a problem-solving approach can be helpful. Allowing students to share ideas and present different solution methods is often insightful.

Table 4.2 – Some Difficulties with Counting Problems
(Adapted from Batanero & Sanchez, 2003)

Difficulty	Explanation	Example
Error in Order	Students may confuse whether or not the order of elements is important (i.e., they might have difficulty distinguishing between combinations and permutations). This may be more difficult if a task isn't a straightforward application, such as the example which requires splitting a set of objects into subsets.	Grace and Jack have four different stickers. In how many ways can they share the stickers if each of them is going to take two of them?
Error of Repetition	Students may confuse when they are allowed to repeat elements.	How many different three digit numbers can be formed with the numerals 1, 2, 3?
Confusing the Type of Object	This error happens when students consider identical objects to be distinguishable or different objects to be undistinguishable.	How many different ways can three identical letters be placed into three of four envelopes – a blue envelope, a green envelope, a red envelope, and a purple envelope?

Exercise 4.6 Solve the examples given in Table 4.2.

Notice that there are four different sampling procedures that we could consider:

1) Order is important and objects are replaced
2) Order is not important and objects are replaced
3) Order is important and objects are not replaced
4) Order is not important and objects are not replaced

These four sampling procedures, together with the fact that objects may or may not be distinguishable can add complexity quite quickly to counting problems. Furthermore, many results can be counterintuitive or illustrate other conceptions, such as those about randomness. However, some tasks that seem complex can be explored using simulations, as illustrated in Examples 4.6 and 4.7.

Example 4.6 – Exploring Counterintuitive Notions

- Suppose you tossed a coin 5 times and obtained the sequence HHHHH. On the next toss are you more likely to obtain an H than a T?

- Lindsay and Mark were each instructed to toss a fair coin 100 times. However, only one of them performed the experiment. The other one just randomly wrote down a sequence of H's and T's. Which person do you think actually performed the experiment based on their sequences below? Explain. (See Essential Question 11.)

<u>Lindsay's Sequence</u>

H, T, H, H, T, H, T, T, H, H, T, H, T, H, H, T, T, H, H, T, H, H, H, T, T, H, H, T, H, T, T, T, H,
H, T, H, H, T, T, H, T, H, T, T, H, H, T, T, H, H, T, T, H, H, T, T, H, T, T, H, T, H, T, T, H, H,
H, T, T, H, T, H, H, T, T, H, T, H, H, H, T, T, H, T, T, H, H, T, H, H, H, T, H, T, H, T, H, H, T,
H

<u>Mark's Sequence</u>

H, H, H, T, T, H, T, H, H, T, H, T, H, H, H, H, H, T, T, H, T, T, T, H, T, T, H, H, T, T, H, T, T,
H, H, T, T, H, T, T, H, H, T, H, H, H, T, T, H, T, H, H, H, T, T, H, H, T, T, T, T, T, T, H, H, T,
H, T, H, T, T, H, H, T, H, T, H, H, H, H, T, H, H, T, T, T, H, T, T, H, T, T, T, T, T, T, T, H, H,
H

Both of these questions highlight counterintuitive notions. The first question highlights two possible fallacies. One is known as the "gambler's fallacy" (or negative recency) and the other as having a "hot hand" (or positive recency). Both fallacies are characterized by the incorrect belief that a random event is affected by another independent event. The gambler's fallacy is a belief that a run of independent events will be broken (in our example, believing that it is more likely that we obtain a T on the next toss than an H since it is about time for a T to occur); whereas, the hot-hand fallacy is the belief that a run of independent events will continue (in our example, believing that we are more likely to obtain an H on the next toss than a T since we have a streak of H's occurring).

In reality, the next toss is independent of the first five tosses. Thus, it is equally likely that the next toss will yield an H as it will yield a T. The gambler might try to argue with the following reasoning. You are tossing a coin 6 times and there are two possible outcomes for each toss (H or T). Hence, by the multiplication principle there are a total of $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$ possible outcomes for the 6 tosses of which HHHHHH can occur in only one possible way. Therefore there is a 1 in 64 chance of obtaining six H's in a row, so the next flip should come up T. Take a moment to think about what is incorrect in this argument. While the counting techniques are sound, the gambler failed to realize that while the probability of obtaining 6 heads in a row on 6 tosses of a fair coin is indeed 1 in 64, that is before we know the results of the first five tosses. Once we know the result of the first five tosses, since the next toss is independent of these, it is just as likely to come up H as T.

Realizing that as the number of tosses of a fair coin increases, the absolute difference between the number of heads and number of tails tends to get bigger; whereas, the ratio of heads to tails tends to 1, can initially be perplexing (Paulos, 2001). The Law of Large Numbers (which is discussed in Section 6) guarantees that the ratio of heads to tails tends to 1 (and most of us expect this – as we toss the coin more, the proportion of heads tends to ½ and the proportion of tails should tend to ½). In terms of absolute difference, just think about the number of tosses. While it is true that as the number of tosses increases we ought to see less variation (also discussed in Section 6), we only need to consider a concrete example to understand this phenomenon – suppose we tossed the coin 100 times and obtained 40% heads and 60% tails, then the absolute difference in the number of heads and tails is 10; whereas, if we tossed the coin 10,000 times and obtained 51% heads and 49% tails, then the absolute difference in the number of heads and tails is 200.

The second example highlights the belief that a sample, no matter its size, should be representative of the characteristics of the parent population. This idea will be discussed further in Section 6. Furthermore, students tend to choose the non-random sequence and cite one or many of the following reasons (Batanero & Sanchez, 2006):

- The random sequence it too irregular;
- There are too many heads or not enough tails (i.e., the belief that the proportion of heads and tails should be very near 50% no matter what the sample size is. To answer the question about the proportion of heads and tails, we should determine a likely proportion to obtain in 100 tosses of a coin. That is, when tossing a coin 100 times, we would expect some variation from 50% heads and 50% tails. This concept will be explored further in Section 6. In this particular example both Lindsay's and Mark's sequences contain close to 50% heads, so students are not likely to use this reasoning for this particular example);
- The runs of a particular outcome (i.e., the runs of heads or tails) are too long.

With all of this said, it turns out that Mark's sequence was randomly generated and Lindsay's sequence was just made up. So, how can you tell? One interesting question has to do with examining the likelihood of a run of heads or tails of a certain length. For example, suppose the results of tossing a fair coin 10 times were H, H, H, T, T, H, T, H, H, T. We might chunk this as follows:

H H H | T T | H | T | H H | T

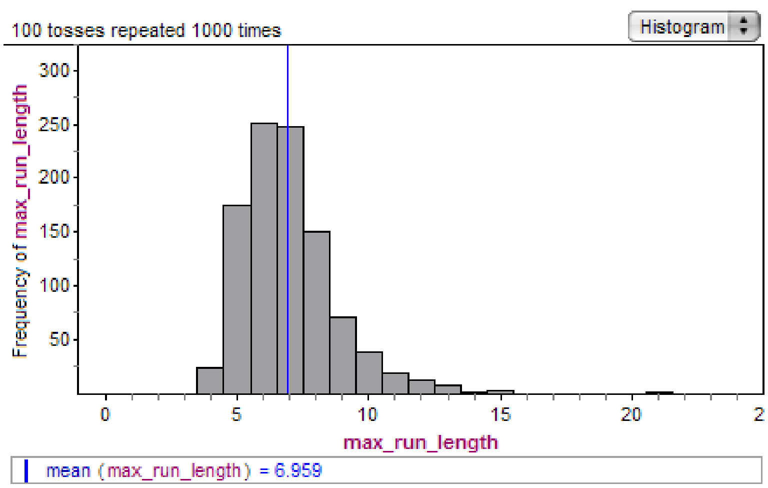
We would say that this sequence of 10 tosses contains 6 runs. We could also specify that there are 3 runs of length 1, 2 runs of length 2, and 1 run of length 3.

While determining the likelihood of a run of a particular length is a difficult question to answer using the concepts of theoretical probability, students can use software programs to begin to explore this idea. There are also interesting questions to informal hypothesis testing that can be explored (informal hypothesis testing is discussed in Section 6). For Lindsay's sequence and Mark's sequence we might begin by making a table to record the number of runs of particular lengths, as shown below.

Mark's Sequence	
Run length	Frequency
1	22
2	19
3	6
4	1
5	1
6	1
7	1
8+	0
Max run length: 7	

Lindsay's Sequence	
Run length	Frequency
1	31
2	27
3	5
4	0
5	0
6	0
7	0
8+	0
Max run length: 3	

We could inspect what the proportion of various runs lengths tends to look like by repeating the experiment (tossing a coin 100 times) many times. For example, repeating this experiment 1000 times and analyzing the distribution of run lengths. However, it is interesting to note that Mark's sequence contains a maximum run length of 7, whereas, Lindsay's sequence doesn't even contain one run of length 4. So, it would be reasonable to explore the distribution of maximum run lengths. That is, if we were to repeat the experiment 1000 times, and record the maximum run in each of these 1000 sequences of 100 tosses, would we be fairly confident that we should have at least one run of length 4? What about a run of length 7? The following histogram shows the results of recording the maximum run length for 100 tosses in each of 1000 trials.



Notice that each of the 1000 trials contained a maximum run of at least 4. This begins to give us convincing evidence that Lindsay's sequence is the one that wasn't generated from tossing the coins. In fact, notice that the average maximum run length is about 7 and that 802 out of 1000 cases, or about 80% of the cases, had at least one run of 6 or greater. In fact, 97.6% of the cases had at least one run of 5 or greater.

An interesting activity is to ask each of the students in your class to toss a coin 100 times and record the sequence of heads and tails. Tell students that you will leave the room and some of them are allowed to cheat. That is, some students are allowed to just randomly write down a sequence of heads and tails while others will actually perform the experiment. The key is that the students are allowed to decide who will cheat and who will actually perform the experiment. They should carefully record this information and keep it from you. Then, the students place their sequences into an envelope with their names on the back or some other way of identifying the sequence to whether or not it was generated by tossing the coin or by 'cheating.' You tell students that you have magical powers and will be pretty successful at picking which students cheated – by looking at the lengths of the maximum runs.

Exercise 4.7 Explore the activity *String of Heads* in Bright, Frierson, Tarr, & Thomas, 2003.

Example 4.7 – The Monty Hall Problem

Consider the following problem.

> You are on a game show where you are given the choice to select one of three doors. Behind one of the doors is a brand new car; while behind the other two doors is a prize that would be worthless to you – a goat. Monty Hall, the game show host, knows which door contains the car. After you initially select a door, Monty opens one of the two doors that you didn't select to reveal a goat. (Notice that Monty can always do this since he knows where the prize is.) Then, Monty gives you the choice of sticking with your original pick or switching to the other remaining door. You need to decide if:
>
> A) It is better to stick with my original door;
> B) It is better to switch to the remaining door; or
> C) It doesn't matter if I stick or switch, it is equally likely that I will win the prize.
>
> Take some time to solve this problem before reading on.

Option C seems to be the most intuitive and most popular answer. However, this answer is not correct. While there are many ways to demonstrate that this isn't the correct answer, we will just look at two approaches.

The first is to make a case table of our various options, and the second is to consider conducting a simulation. Let's begin with a case table – listing all the possible ways the goats and car can be arranged behind the doors.

	Door 1	Door 2	Door 3
Case 1	Car	Goat	Goat
Case 2	Goat	Car	Goat
Case 3	Goat	Goat	Car

Let's assume that you initially picked Door 2 and analyze each case.

- If the prizes were arranged as in Case 1, Monty would open up Door 3. If you stayed with your original pick you would lose; whereas, if you switched you would win.

- If the prizes were arranged as in Case 2, Monty would open either Door 1 or Door 3. In either case, if you stayed with your original pick you would win; whereas, if you switched you would lose.

- If the prizes were arranged as in Case 3, Monty would open Door 1. If you stayed with your original pick you would lose; whereas, if you switched you would win.

Notice, that in two of the three cases, if you switch from your original pick you would win. You can verify that this is true no matter which door you originally picked. Therefore, if you switch from your original pick you have a 2/3 chance of winning, and if you stay with your original pick you have a 1/3 chance of winning.

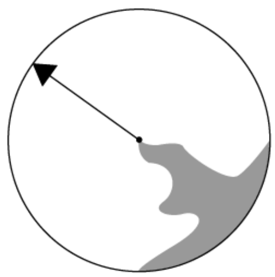
Exercise 4.8 If you are still not convinced that it is beneficial to switch doors from your original pick, design a simulation to determine the probability of sticking versus switching. Then, explore the simulation *Stick or Switch* from the National Library of Virtual Manipulatives at http://nlvm.usu.edu/en/nav/category_g_3_t_5.html.

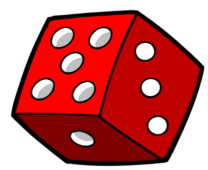
This example represents another situation where the results can seem counterintuitive. The situation is modeled after the game show *Let's Make a Deal*. The problem was posed to Marilyn vos Savant, author of the "Ask Marilyn" column in *Parade Magazine*. Her solution caused quite the uproar in the mathematics community – partially due to the counterintuitive result, and partially due to disagreement in the assumptions based on the problem statement. For some links that point you to some discussions on the problem and the controversy, go to http://math.rice.edu/~pcmi/mathlinks/montyurl.html. This problem highlights the difficulty of the concept of independence (independence will be discussed further in Section 6).

The previous problems illustrate how counterintuitive results can happen when attempting probability problems. Furthermore, counting problems and probability problems can get quite complex fairly quickly. In practice, performing experiments and running simulations can be a way to begin to understand complex problems. The next sections will discuss many of the conceptions that students hold around concepts of probability, along with some teaching strategies.

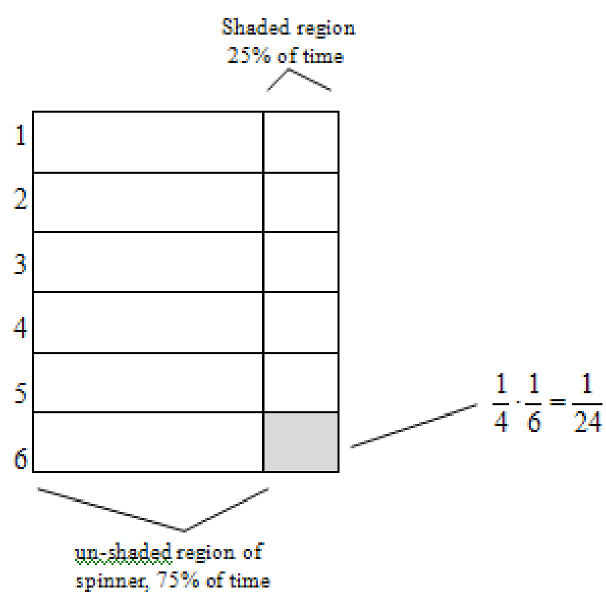
Example 4.8 – Area Models

While we can determine the theoretical probability of an event by listing all the possible outcomes and determining the number of ways in which the event can occur, when dealing with experiments with two or more independent events this task can become quite difficult. Furthermore, if the events are not all equally likely this can complicate the task. Often an area model can be used to help visualize the solution. As an example, suppose you are going to spin the arrow on the spinner from Essential Question 2 in Section 3 one time and also toss a six-sided number cube (with the numerals 1, 2, 3, 4, 5, 6 on the sides with each occurring once). And, you want to determine the probability that the arrow will land in the shaded region and the number cube will land with a 6 facing up.



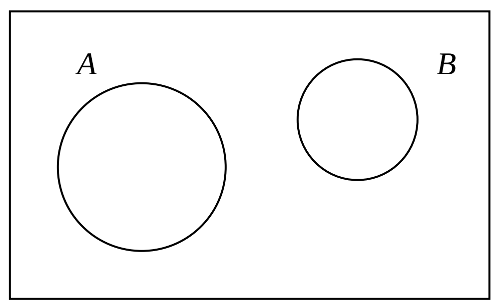


From performing the experiment earlier, you have decided that the arrow on the spinner lands in the shaded region 25% of the time. Then, the following area model can be used to model the probability of landing in the shaded region and obtaining a 6 (the rectangle represents a whole and it has been partitioned in one direction to represent spinning the spinner and landing on the shaded region 25% of the time, and partitioned in the other direction to represent the six equally likely outcomes of the die).



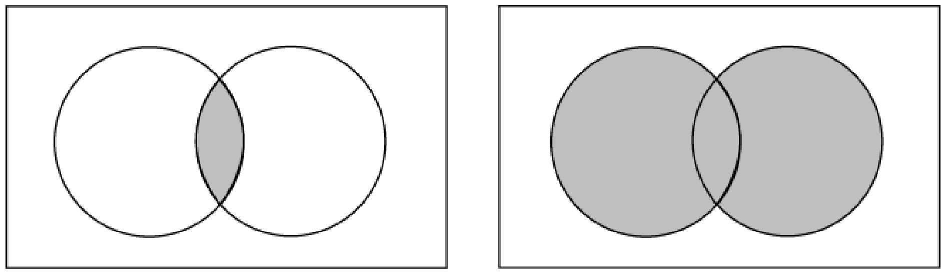
In addition to the key definitions and results in Table 4.1, there are some key results from probability that deal with mutually exclusive events and independent events that should be recalled, as outlined in Table 4.3. These key results will be illustrated in examples following Table 4.3.

Table 4.3 – Some Key Results from Probability

Result	**Description**
Mutually Exclusive Events	Mutually exclusive (or disjoint) events are events that can not occur at the same time. Often Venn Diagrams are used to show a visual representation of the relationships between sets. For example, if we considered the rectangle below to represent the sample space for an experiment and the circle labeled *A* to represent Event *A* and the circle labeled *B* to represent Event *B*, then, visually, *A* and *B* being mutually exclusive is depicted by the circles not overlapping (i.e., the intersection of the two sets is empty). Note, when dealing with Venn Diagrams, it is often the case that the areas of the circles representing the various sets are sketched and not proportional to the size of the sets. 
Independent Events	Two events, *A* and *B*, are said to be independent if the probability of one of the events occurring is not affected by the other event occurring or not occurring. If *A* and *B* are independent events, then $P(A \text{ and } B) = P(A) \cdot P(B)$.
Addition Rule	For events *A* and *B*, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.
Conditional Probability	For events *A* and *B*, where the probability of *B* occurring is greater than 0, the probability that *A* occurs given that *B* has occurred, denoted by $P(A \mid B)$, is given by $P(A \mid B) = \dfrac{P(A \text{ and } B)}{P(B)}$.

Example 4.9 – Some Notes about Venn Diagrams

Venn Diagrams can be useful for motivating the results shown in Table 4.3. Recall that the intersection of two events, *A* and *B*, and the union of two events, *A* and *B*, can be shown using Venn Diagrams as follows. [Note: the intersection refers to both *A* and *B* occurring, while the union refers to *A* occurring, *B* occurring, or both *A* and *B* occurring.]



Representing *A* and *B* occurring Representing *A* or *B* occurring

Notice that if *A* and *B* are not mutually exclusive, then their intersection is not zero. Therefore, to find the probability that *A* or *B* occurs, we would need to add the probability that *A* occurs to the probability that *B* occurs and then subtract from this sum the probability that *A* and *B* occurs. We subtract this probability since it is included twice (i.e., once in the probability of *A* occurring and once in the probability of *B* occurring). If *A* and *B* are mutually exclusive, then we have a situation like that represented in Table 4.3. Thus, the probability of *A* and *B* is zero and the addition rule reduces to $P(A \text{ or } B) = P(A) + P(B)$.

Notice, that if we solve the conditional probability rule for $P(A \text{ and } B)$ we obtain:

$$P(A \text{ and } B) = P(B) \cdot P(A \mid B) .$$

In general, the probability of *A* and *B* occurring can be determined from this result. If *A* and *B* are independent, then the probability of one of the events occurring is not affected by the other event occurring or not occurring. Hence, $P(A \mid B) = P(A)$ (or equivalently, $P(B \mid A) = P(B)$), and we find for independent events:

$$P(A \text{ and } B) = P(B) \cdot P(A)$$

The conditional probability result can be motivated by a Venn diagram. That is, if we are looking for the probability that event *A* occurs given that event *B* occurs, we simply focus our attention on *B* as the underlying sample space. That is, we are looking for the portion of *A* that is within *B* – which is exactly the ratio given by $\dfrac{P(A \text{ and } B)}{P(B)}$.

In general, a Venn Diagram can **not** be used to determine whether or not two events are independent. Also, notice that mutually exclusive and independence are two different things. Take a moment to think about this and record your thoughts in your journal.

Exercise 4.9 Is $P(A \mid B)$ always equal to $P(B \mid A)$? Explain.

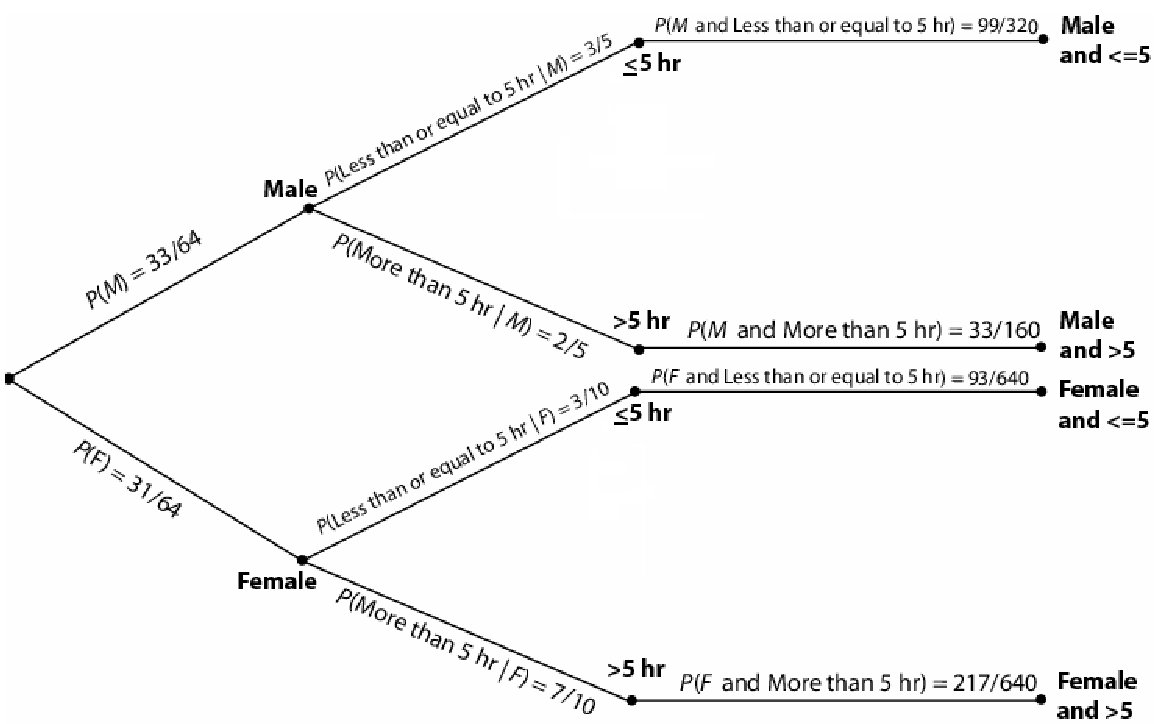
Examples 4.10 and 4.11 serve to illustrate some of the concepts in Table 4.3.

Example 4.10 – Conditional Probability and Various Representations

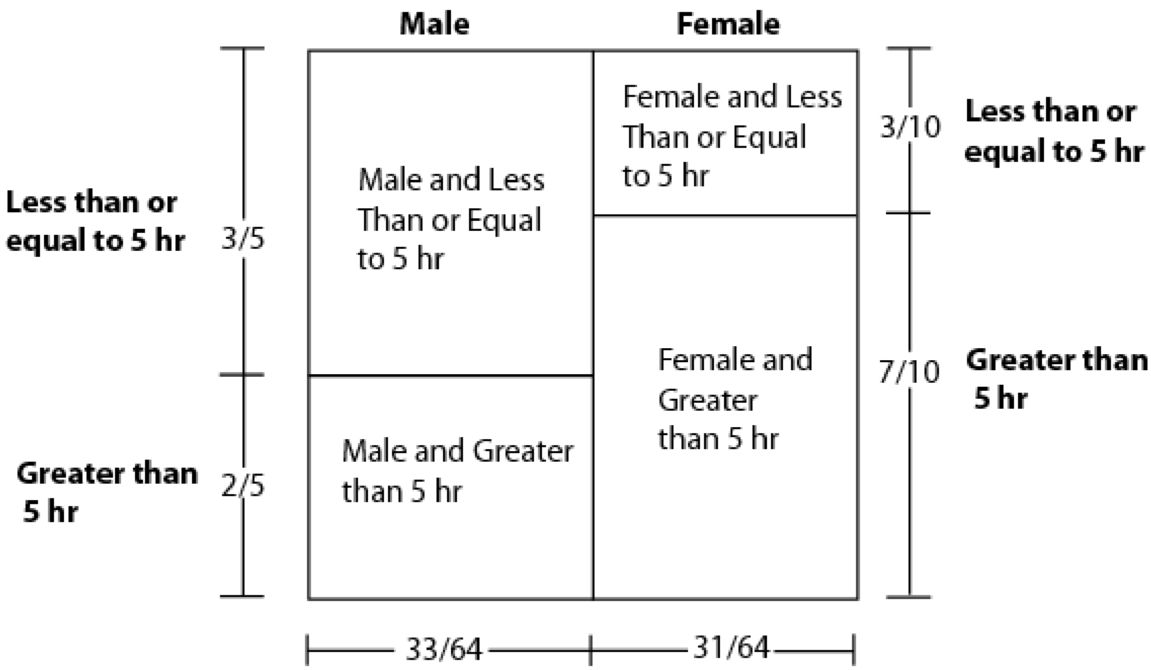
On a statewide assessment, 16,000 students were asked on average how many hours they spend on mathematics homework in a week. This table shows the results disaggregated by gender.

	Less than or Equal to 5 Hours	**More than 5 Hours**	**Total**
Male	4950	3300	8250
Female	2325	5425	7750
Total	7275	8725	16,000

Notice that the categories of Male and Female are mutually exclusive and together comprise the entire sample space. Additionally, the categories "Less than or Equal to 5 Hours" and "More than 5 Hours" are mutually exclusive. The information from this table might be organized into a tree diagram, showing various probabilities as follows.



Or, the information may be organized into an area model as shown below.



Exercise 4.10 Given that a student is male, what is the probability that the student spends less than 5 hours a week on homework?

Exercise 4.11 Explain how to determine the answer to Exercise 4.10 by using each of the three models above.

Exercise 4.12 Explain how to determine the probability that a student is female using each of the different models above.

Exercise 4.13 Determine the probability that a student is female and spends more than 5 hours a week on homework. Explain which model you used and how you determined your answer.

Example 4.11 – Independent or Not
(Adapted from Watkins, Scheaffer, & Cobb, 2008)

Rich and Alisa conducted a taste test with 100 adults to see how they would perform in identifying bottled water versus tap water. The following two tables summarize some of their results. The first table shows how those surveyed performed based on whether or not they regularly drink bottled water; whereas, the second table shows how people performed based upon their gender.

	Did the person correctly identify the tap water?			
Do you drink bottled water regularly?		**Yes**	**No**	**Total**
	Yes	24	6	30
	No	36	34	70
	Total	60	40	100

	Did the person correctly identify the tap water?			
Gender		**Yes**	**No**	**Total**
	Male	21	14	35
	Female	39	26	65
	Total	60	40	100

Based on these data, take a moment to determine whether or not the events *drinks bottled water regularly* and *correctly identifies tap water* are independent or dependent events. Record your thoughts in your journal. Repeat this exercise for the events *is a male* and *correctly identifies tap water*.

To determine whether or not particular events are independent, we just need to utilize the information in Table 4.3. Let's begin by letting *A* be the event *drinks bottled water regularly*, *B* be the event *correctly identifies tap water*, and *C* be the event *is a male*.

Suppose we selected one of the 100 adults who participated in the taste test at random. If the events *A* = *drinks bottled water regularly* and *B* = *correctly identifies tap water* are independent, then we would have $P(B \mid A) = P(B)$. That is, correctly identifying tap water would not depend upon whether or not a person drinks bottled water regularly. From the first table, we can see that $P(B \mid A) = \frac{24}{30}$ by looking across the first row, and that $P(B) = \frac{60}{100}$. So, the two events must be dependent upon each other since these probabilities are not the same. Whereas, if we look at the probability that the person correctly identified tap water given that the person is male, or $P(B \mid C)$, we find by looking across the first row of the second table that $P(B \mid C) = \frac{21}{35} = \frac{60}{100}$. Thus, gender does not seem to be a factor in correctly identifying tap water.

In general, it is good practice to **not** assume independence. That is, if a problem doesn't explicitly state that events are independent, you should always check.

Sections 5 and 6 will focus on examining students' conceptions about probability, and focus on teaching strategies. In particular, students' conceptions about independence and conditionals will be discussed in Section 6. Again, if you wish to have a more in-depth content review of the topics discussed in this section, it is recommended that you explore your curricular materials or the references in Section 10.

Section 5: Students’ Understandings of Probability

This section examines some typical student misconceptions and understandings around probability and illustrates those misconceptions with examples. By no means does this section cover all the misconceptions that students have, but we illustrate some of the most typical ones. We will also revisit many of the Essential Questions from Section 3 and tie them to these misconceptions. The next section will examine some instructional strategies that can help teachers foster middle students’ understandings of probability and some of the challenges described in this section.

Initial research into students’ understandings of probability concepts was conducted during the 1950s and 1960s by Piaget and Inhelder, along with various psychologists. These researchers were not motivated by an interest in probability as part of a school curriculum, but rather by the developmental growth of adults and children in regard to their thinking about probability. However, their work inspired researchers to study the heuristics (strategies) that people use when facing probability tasks. Over the last fifteen years, curriculum reform has integrated probability and statistics into the curriculum and has resulted in research regarding the learning and teaching of probability. This current research has focused on the learning and teaching of probability in various age ranges (i.e., elementary school, middle school, and secondary school) and over a wide range of topics. (Jones & Thornton, 2005)

Table 5.1 summarizes some of the various conceptions that students hold that researchers have studied. Examples of these conceptions will be presented throughout this section.

Table 5.1 – Typical Student Difficulties and Conceptions with Probability

Students’ Conceptions	Description
The Ratio Concept	The ratio concept refers to students’ conceptions about probability as a ratio. This conception requires an understanding of part-to-whole relationships and that the probability of an event occurring is the number of outcomes in that event divided by the number of possible equally likely outcomes.
Equiprobability Bias	LeCourtre and her colleagues have documented the Equiprobability bias which refers to children’s conception that all outcomes from a probability experiment have an equal chance of occurring (Shaughnessy, 2003)
The Representativeness Heuristic	The representative heuristic refers to evaluating probabilities of an event based on how well that event is seen to represent the perceived parent population or the process by which the event is generated.

Table 5.1 Continued…

The Availability Heuristic	The availability heuristic refers to estimating the likelihood of an event based upon one's experience with the occurrence of that particular event.
The Outcome Approach	The outcome approach is a term identified by Konold to describe how students respond to probability tasks where they do not view the results of a single trial of an experiment as one of many possible outcomes (Shaughnessy, 2003).
Conjunctions and Conditionals	There has been quite a bit of research regarding students' difficulties in dealing with conditionals (the probability of an event *B* happening given that an event *A* has happened where *A* and *B* are not independent). Additionally, students tend to have difficulty identifying whether or not the events are independent. In particular, an area where conditional events seem to be problematic is when the conditioning event occurs after the event that it conditions (Shaughnessy & Bergman, 1993) and an area where conjunctions seem to be problematic is known as the conjunction fallacy – believing that the probability of events *A* and *B* occurring is greater than the probability of one of those events occurring (Shaughnessy, 2003).

Example 5.1 – The Ratio Concept

Consider the following problem given by David Green in a study of over 3000 students ages 11-16 in Great Britain (Shaughnessy, 2003).

Two bags have black and white counters.

Bag A: 3 black and 1 white
Bag B: 6 black and 2 white

Which bag gives the better chance of picking a black counter?

A) Same chance
B) Bag A
C) Bag B
D) Don't know

Why?

More than 50% of the students in the study choose Bag B and 39% of all of the students gave the reason that Bag B contains more black counters (Shaughnessy, 2003).

As we can see, an understanding of part-whole relationships is fundamental when working with probability tasks.

Reporters of the 20/20 ABC newsmagazine examined the ratio concept in an experiment that aired on 2/23/07. Adults and children were presented with two plates of jelly beans. Each plate contained some red jelly beans and some white jelly beans. The first plate contained more red jelly beans than the second plate; however, the ratio of red to white jelly beans was less for the first plate than the second plate. Researches asked adults and children which plate they would choose if they wanted a better chance of randomly selecting a red jelly bean. About 33% of those surveyed choose the first plate.

Exercise 5.1

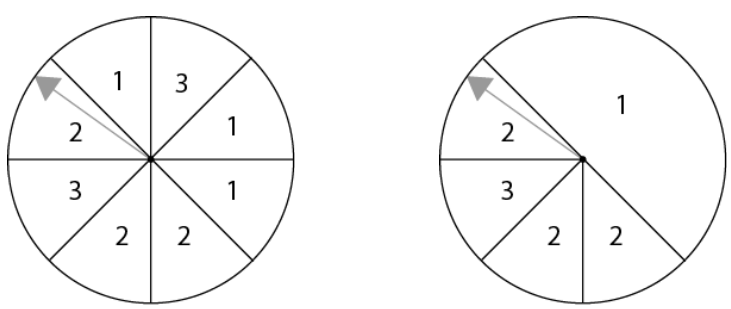
Explain how the ratio concept connects to the ideas in M:N&O:7:1.

M:N&O:7:1 **Demonstrates conceptual understanding of rational numbers with respect to** percents as a means of comparing the same or different parts of the whole when the wholes vary in magnitude (e.g., 8 girls in a classroom of 16 students compared to 8 girls in a classroom of 20 students, or 20% of 400 compared to 50% of 100); and percents as a way of expressing multiples of a number (e.g., 200% of 50) **using models, explanations, or other representations**.

Exercise 5.2

Consider the following question.

Tom spins the arrow on each spinner below once. Which arrow is more likely to land on a section numbered 1 – Spinner A or Spinner B?



Spinner A **Spinner B**

Explain how students who struggle with the ratio concept might respond to the above question.

Example 5.2 – Equiprobability Bias

Consider rolling two standard six-sided number cubes and recording the sum of the numbers that land face up. Students exhibiting the equiprobability bias will believe that the possible sums are equally likely.

Example 5.3 – The Representativeness Heuristic

Take a moment to revisit the first part of Essential Question 1 from Section 3.

> **Essential Question 1** A fair coin is tossed six times and the results are recorded in the order they appear. At each toss, the coin lands either H (heads) or T (tails).
>
> The outcome H T H T T H
>
> ☐ is less likely than
> ☐ is as likely as
> ☐ is more likely than
>
> the outcome H H H H H H.
>
> Explain your reasoning.
>
> (Adapted from Callaert; 2002)

Students exhibiting the representativeness heuristic believe that the outcome HTHTTH is more likely to occur than the outcome HHHHHH. This is most likely due to their perceived notion that HTHTTH is more representative of the 50-50 expected ratio of heads to tails when flipping a fair coin. Students believe that HHHHHH is not representative of the random process of flipping a coin. However, the sample space of flipping a fair coin 6 times consists of 64 equally likely outcomes (there are two choices for each flip – heads or tails – hence a total of $2^6 = 64$ equally likely outcomes by the multiplication principle). And, HTHTTH and HHHHHH are each an outcome in this sample space and hence are equally likely to occur. Notice how there is a subtlety in this question. We are not talking about the number of heads that would be likely to occur in 6 tosses of a fair coin.

Exercise 5.3 Go back and revisit Example 4.6 and explain how the task in the first bullet in that exercise connects to the representativeness heuristic.

Example 5.4 – The Representativeness Heuristic

Another example where students rely on representativeness is with regards to problems dealing with sample size. Representativeness often causes students to neglect the sample size. For example, reconsider Essential Question 4 from Section 3.

> **Essential Question 4** Assume that the chance of having a boy or girl baby is the same. Over the course of a year, in which type of hospital would you expect there to be more days on which at least 60% of the babies born were boys? Explain your reasoning.
>
> A) In a large hospital
> B) In a small hospital
> C) It makes no difference
>
> (Adapted from Shaughnessy & Bergman, 1993)

It should be noted that the magnitude of the difference is important. Here we are assuming that the magnitude of the difference is sufficiently large. For example, if the "small" hospital had on average 2 babies born a day and the "large" hospital 6, you would expect there to be more days on which at least 60% of babies born were boys to occur at the "large" hospital.

In a study conduct by Kahneman and Tversky of college freshman who hadn't studied probability and statistics, the majority of students choose option C. These results were replicated in a study by Shaughnessy, where students responded that they choose option C since the chance of obtaining a boy was the same in any hospital. (Shaughnessy & Bergman, 1993) In this case, students tend to be relying on the 50-50 aspect of a baby being born either a boy or a girl and completely ignoring the sample size. That is, it is more likely in a small hospital that there will be more days over the course of a year in which at least 60% of the babies born were boys (e.g., if only 5 babies are born on average a day in the small hospital, it wouldn't be unreasonable that 3 of those 5 babies are boys; as the sample size increases obtaining at least 60% boys becomes more unlikely – take a moment to reflect on the distribution). This question will be revisited in Section 6.

Exercise 5.4 Revisit Essential Question 5 from Section 3.

> **Essential Question 5** There are two cab companies that operate in the city, a Blue Cab company and a Green Cab company. It is known that 85% of the cabs in the city are Green and 15% are Blue. A cab was involved in a hit-and-run accident at night. A witness at the scene identified the cab involved in the accident as a Blue cab. This witness was tested under similar visibility conditions, and made correct color identifications in 80% of the trial instances. Given that the witness identified the cab as blue, what is the probability that the cab involved in the accident was a Blue cab rather than a Green one? Explain your reasoning.
>
> (Adapted from Shaughnessy & Bergman, 1993)

Explain how representativeness may affect students' reasoning on this problem. (Note: this problem will be revisited in Example 6.5.)

Example 5.5 – The Availability Heuristic

The following is an example of relying on availability when making a judgment about the likelihood of an event.

Karen had a few of her friends drop out of high school so she perceived that the drop-out rate at the school was increasing when in reality it was decreasing.

Example 5.6 – The Availability Heuristic

Availability also tends to manifest itself when asking about counting problems. For example, problems that are perceived to be more complex are often seen to contain a greater number of possible arrangements. Consider the following question.

> Compare the number of ways to form a three person committee from a selection of 12 people with the number of ways to form a nine person committee from a selection of 12 people. (Adapted from Shaughnessy & Bergman, 1993)

It is often perceived that it is a more complex task to form a nine person committee than a three person committee and hence there should be more ways to form a nine person committee from 12 people, even though there are 220 ways for each of these (take a moment to verify this).

While representativeness and availability can result in faulty probabilistic reasoning, they can also be useful in various situations. For example, when selecting a sample from a population, we try to ensure that the selection is done in a random way to ensure that it is representative of the population (Shaughnessy & Bergman, 1993). Teachers need to identify when students are using heuristics in such a way as to lead to misconceptions versus when heuristics are being properly used to understand various situations involving probability and statistics.

Example 5.7 – The Outcome Approach

In an experiment by Shaughnessy, Watson, Moritz, & Reading, middle school students were asked to predict how many red chips would be pulled in an handful of 10 red chips from a bag containing 50 red chips, 30 blue chips, and 20 yellow chips. Students were then asked to predict what would happen if this experiment was repeated six times. Students who considered the proportion of red chips in the bag tended to give intervals that clustered around 5; whereas, students who exhibited the outcome approach tended to give unusually large intervals or wrote intervals such as "1, 3, 5, 7, 9, 10" and explained that anything could happen. (Shaughnessy, 2003)

Example 5.8 – Connecting to Proportionality

The previous example provides an opportunity to design a task that elicits students understanding of proportional reasoning and its connection to probability and statistics (also see *Proportional Reasoning – A Research Based Unit of Study for Middle School Teachers*). Proportional reasoning is a major topic in middle school mathematics. And, researchers emphasize the need for more students to recognize when proportional reasoning is appropriate throughout the curriculum. As an example, Watson and Shaughnessy have studied students' use of proportional reasoning in data and chance. In fact, the task explained in Example 5.7 is a modified 1996 NAEP task. The researchers felt that this task restricted what they could learn about students' use of proportional reasoning, since the original task only asked students to predict the results of one sample (Watson & Shaughnessy, 2004). Hence, the task was slightly amended to ask students to predict what would happen if the experiment was repeated 6 times, and subsequently what would happen if the experiment was repeated 40 times. They categorized students' responses into three categories:

- *Students who predicted too low* – these students did not use proportional reasoning, and often reasoned that half of the chips are not red, so you wouldn't expect many to be red; or, since there are three colors, about 1/3 should be red;
- *Students who predicted too high* – these students tended to reason that there should be at least 5 reds and sometimes rejected the clustering around 5 even after repeating the experiment; this reasoning may be intermediate to proportional reasoning, and may suggest the need to repeat the experiment more times;
- *Students who reasoned proportionally* – these students were more likely to suggest a reasonable variation for the number of reds obtained on six trials.

Exercise 5.5 What additional questions might you ask students to test their understanding of proportional reasoning around the task described in Examples 5.7 and 5.8?

Exercise 5.6 Describe some other tasks that might be used to assess students understanding of the role of proportionality in data and statistics.

Exercise 5.7 Examine *Task 1 – Comparing Two Data Sets* from Watson & Shaughnessy, 2004 and discuss the connection to proportional reasoning.

Example 5.9 – The Outcome Approach

When asked for the probability of some event, students using the outcome approach view their task as predicating the result of a single trial rather than the distribution of occurrences in a sample (Konold, 1995). For example, consider Essential Question 8 from Section 3.

> **Essential Question 8** The Sutton Meteorological Center wanted to determine the accuracy of the weather forecasts. They searched their records for those days when the forecaster had reported a 70% chance of rain. They compared these forecasts to records of whether or not it actually rained on those particular days.
>
> The forecast of 70% chance of rain can be considered very accurate if it rained on:

a) 95% - 100% of those days.
b) 85% - 94% of those days.
c) 75% - 84% of those days.
d) 65% - 74% of those days.
e) 55% - 64% of those days.

Explain your reasoning.

(Adapted from Konold, 1995)

Konold and Garfield have administered this item to a number of college students at the beginning of a statistics course. Roughly 36% of students (n = 119) choose option a) as opposed to 32% of students who choose option d) (the correct option). (Konold, 1995) Students who chose option a) expect rain to occur almost all of the time when a forecaster predicts a 70% chance of rain. Thus, if the forecaster were to predict a 90% chance of rain, it is likely that students wouldn't view this as a greater chance of obtaining rain.

Example 5.10 – The Conditioning Event occurs after the Event it Conditions

Consider the following problem by Falk (adapted from Shaughnessy & Bergman, 1993). Take a moment to solve the problem before reading on.

> A bag contains two white marbles and two black marbles. A marble is drawn from the bag and **not** replaced. Then a second marble is drawn from the bag.
>
> a) What is the probability that the second marble is white given that the first marble is white?
>
> b) What is the probability that the first marble is white, given that the second marble is white?

Notice that part a) is asking us to find P(second marble is white | first marble is white); whereas, part b) is asking us to find P(first marble is white | second marble is white).

Part a) is usually fairly straightforward for students who have studied some conditional probability. They can determine that after the first draw there are 3 marbles left in the bag, of which 2 are black and 1 is white. Therefore, the probability that the second marble is white given that the first marble is white is $\frac{1}{3}$.

However, part b) is often seen as impossible by students, since they believe that the outcome of the first draw can not depend upon the outcome of the second draw. Falk believes this difficulty is due to causation. (Shaughnessy & Bergman, 1993) That is, in part a), students can infer cause for the second marble being white, but in the second case, they can not infer cause for the first event from the second event.

One can see that the probability that the first marble is white given that the second marble is white is also $\frac{1}{3}$ by writing out the set of all possible outcomes. In this case, it may be easier for students if they number the marbles in the bag to help distinguish among them. That is, if they call the first white marble W_1, the second white marble W_2, and so on for the black marbles. Then, writing out the sample space we obtain 12 equally likely outcomes:

$$\{W_1W_2, W_1B_1, W_1B_2, W_2W_1, W_2B_1, W_2B_2, B_1B_2, B_1W_1, B_1W_2, B_2B_1, B_2W_1, B_2W_2\}$$

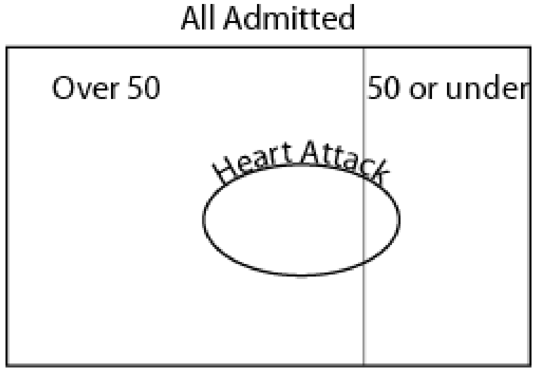
Furthermore, we can see that if we consider only the outcomes where the second marble is white (6 of them), we find that in 2 of these the first marble is also white. Therefore, the probability that the first marble is white given that the second marble is white is $\frac{1}{3}$.

Example 5.11 – The Conjunction Fallacy

Take a moment to solve the following problem before reading on.

> On a particular day, a very large hospital monitors all the patients who are admitted. Which of the following is more likely?
>
> A) A person is admitted for a heart attack.
>
> B) A person is admitted for a heart attack and is over the age of 50.

Many people think that option B is more likely. Perhaps this is due to associating people over the age of 50 with having heart attacks more often than younger people (assuming that we are given that a person has had a heart attack and are being asked to compare the likelihood that the person is over 50 to the likelihood that the person is 50 or under – a completely different question). However, on a particular day, the number of people who were admitted for heart attacks must be greater than or equal to the number of people who were admitted for heart attacks and were over the age of 50 (e.g., all that is needed for the first set to exceed the second set is one person under the age of 50 being admitted for a heart attack). An area model may help to see this relationship. For the problem above, a reasonable area model might look something like the following.

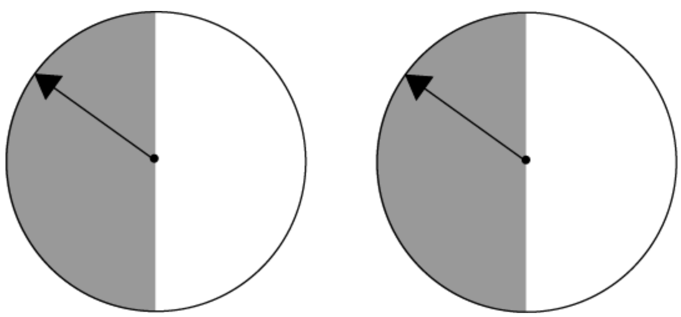


The diagram shows that those who are admitted are either over 50 or 50 or under (that is, these two sets are disjoint). When we are asked to compare the likelihood of being admitted for a heart attack to being admitted for a heart attack and being over 50, we are comparing the area of the entire ellipse to the area of the ellipse which is to the left of the vertical line, respectively. Hence, we can see that A) is more likely than B).

We can also see that if we assume that the person had a heart attack (represented by the area of the ellipse), it is more likely that the person was over 50 than 50 or under (the area in the ellipse to the left of the vertical line compared to the area within the ellipse to the right of the vertical line). Researchers have also studied whether or not the conjunction fallacy is dependent upon the context of the problem. That is, in the previous problem, perhaps it was the context of a heart attack that leads one to believe that it would be more likely that a person was admitted for a heart attack and over the age of 50. Zawojewski and Shaughnessy examined student work from Essential Question 3 from Section 3, which is absent of context.

Essential Question 3

Alisa said that if each arrow on each spinner below is spun simultaneously there is a 50% chance that both arrows will land on the shaded-gray section. Explain whether or not you agree with Alisa.



(Adapted from Mitchell et al., 1999)

Only 8% of the students who took this item on the 1996 twelfth grade NAEP assessment disagreed with the statement and gave a correct explanation. An additional 20% disagreed with the statement and gave a partial explanation. (Mitchell et al., 1999). When Zawojewski and Shaughnessy examined a convenience sample of student responses, they found that students' difficulties with conjunctions were more than just psychological and students lacked the ability to list all of the possible outcomes (Shaughnessy, 2003). See Table 6.2.

Example 5.12 – Conditionals in Social Situations

Researchers have also asked students about conditional probability within the context of social situations. For example, consider the following problem (Watson, 2005).

> What is the probability that a women (W) is a school teacher (T) (i.e., $P(T \mid W)$) and what is the probability that a school teacher is a women (i.e., $P(W \mid T)$)?

Take a moment to think about a reasonable answer to this problem before reading on.

These types of questions allow researches to determine if students can distinguish these probabilities from each other but also from $P(W)$ and $P(T)$. A reasonable answer to the question above would be one where $P(T \mid W) < P(W \mid T)$.

Students are sometimes asked about a problem where statements were placed in a probability setting (e.g., what is the probability…) and sometimes when statements where placed in a frequency setting (e.g., how many…). Students tended to perform better in the latter situation (Watson, 2005). Perhaps this can give us some insight in teaching conditionals.

Now that we have a better understanding of students' conceptions of probability, the next section will discuss some strategies for teaching probability concepts and will focus on fostering middle school students' understandings of compound events and conditional probability and independence.

Section 6: Teaching Probability for Understanding

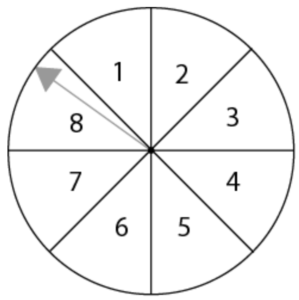
The previous section covered many of the conceptions that students have about determining the probability of some event. While we examined many of the possible misconceptions that students hold, and this list may be daunting, researchers have also offered some suggestions for teaching probability in middle school for understanding. This section will examine some of those suggestions. Furthermore, as the three principles of learning suggest (see Table 1.1), we should seek to engage students' preconceptions and use these preconceptions to help build new knowledge and connect procedural understanding and conceptual understanding.

We begin by looking at some of the characteristics of random experiments that are recognized by children around the age of 10 or 11, as identified by researchers. These characteristics are summarized in Table 6.1 (Pratt, 2005).

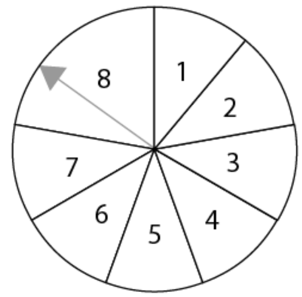
Table 6.1 – Characteristics of Randomness Understood by Young Children (as described by Pratt, 2005)

Characteristic	Description
Unpredictability	If the next outcome of an experiment is not predictable, then the child might consider the experiment to be random.
Irregularity	If a child looks at the outcomes of an experiment that is repeated over time, and this sequence of outcomes does not seem to exhibit a pattern, then the child might consider the experiment to be random.
Unsteerability	If the child is unable to exhibit physical control over the experiment, then the child might consider the experiment to be random.
Fairness	If the experiment exhibits rough symmetry, then the child might consider the experiment to be random.

Pratt claims that these intuitions held by young children are not much different than those held by an expert – the difference being that an expert makes connections among these ideas and understands the difference between random and fairness. For example, a child might see Spinner A and Spinner B and regard Spinner A as being fair and thus random, but might also regard Spinner B as being unfair and hence not random. (Pratt, 2005)



Spinner A



Spinner B

In addition an expert understands the Law of Large Numbers. That is, a random experiment may be unpredictable in the short term, but over the long run the relative frequency of an outcome approaches its theoretical probability (or that sample proportions converge to the true probabilities in random sampling as the sample size increases). The good news is that strategies exist to help students in middle school understand key aspects of probability as illustrated in Table 6.2.

Table 6.2 – Strategies for Teaching

Strategy	Description
Testing Personal Conjectures	As seen in the previous section, students may have many misconceptions around probability concepts. As Pratt points out, misconceptions are often seeing as needing eradication (Pratt, 2005). However, as the second principle in Table 1.1 highlights, and as Pratt notes, misconceptions can become learning opportunities. As Table 6.1 highlights, students often see randomness as unpredictable, and this can be a starting point for building new ideas. For instance, helping students understand that randomness can be predictable in the long run. However, if students are going to build new knowledge from their current beliefs, those beliefs need to be challenged but in such a way that it convinces the children that their current belief structures have weaknesses. Allowing students to test their personal conjectures is crucial to this development.

Table 6.2 Continued…

Large Scale Experiments	Ideas like the Law of Large Numbers (which will be explored in this section) require large scale experiments. Tasks should be designed to encourage students to use a large number of trials (Pratt, 2005).
Introduction at a Young Age	Researchers have shown that students develop some of the concepts and skills needed to understanding ideas about chance at an early age (Shaughnessy, 2003). The National Council of Teachers of Mathematics recommends an increased emphasis on probability and statistics. Furthermore, this increased emphasis is intended to span the grades, rather than being reserved solely for middle school and high school (National Council of Teachers of Mathematics, 2000).
Emphasize Sample Space	Students tend to be weak in the idea of developing a sample space, as illustrated in the NAEP item in Example 6.11. Students tend to disagree on the set of all possible outcomes (Shaughnessy, 2003), but this can be a teachable moment and lead to a discussion of how to represent the outcomes in various ways and which ways show equally likely outcomes (e.g., when presented with a bag that contains only green marbles and white marbles, a student may list the sample space as {green, white}; however, these outcomes may not be equally likely – there may be 3 green marbles and 4 white marbles in the bag, leading another student to list the sample space as {green 1, green 2, green 3, white 1, white 2, white 3, white 4})

Table 6.2 Continued…

Connect Probability and Statistics	We will revisit this idea further in this section, but you may want to take a moment to look back at Essential Question 12 from Section 3. Asking questions which make connections between the sample space in probability and the nature of variation in statistics helps students see the range of likely outcomes and informally develop the idea of confidence intervals (Shaughnessy, 2003)
Introduction through Data	Previously, researches have recommended that students begin with probability experiments, conduct simulations to gather data, and then answer questions about some probability conjecture (Shaughnessy, 2003). And, while many of the examples throughout this unit follow this style (e.g., the spinning pennies experiment), recent research suggests that it may be beneficial to introduce students to ideas about probability through data (Shaughnessy, 2003). That is, start with actual data and ask students probability questions. This does not mean that the probability experiments mentioned earlier are not important. It simply implies, that as a starting point, it may be beneficial to introduce probability through data (e.g., similar to Essential Question 5 from Section 3 – however, you may want to use more basic questions for introduction).

Table 6.2 Continued…

Problem-Solving	To do statistics means that one formulates a well-posed question (i.e., develops a hypothesis), makes decisions on how to gather data, and then chooses to display that data with a particular representation with the goal of either accepting or rejecting that original hypothesis. Researchers recommend that students are given the opportunity to explore probability on their own and conduct their own projects (Shaughnessy, 2003). This is consistent with our general philosophy that students should be given the opportunity to build and construct knowledge. The nature of probability and statistics lends itself to this approach.

Examples 6.2 – 6.5 serve to illustrate the strategies in Table 6.2 (the ideas of emphasizing a problem-solving approach will be integrated throughout the examples). Following these examples, we will look to extend these ideas to include building an informal understanding of hypothesis testing at the middle grades (a formal study of hypothesis testing is reserved for a high school statistics course). Then, we will close the section by examining strategies for teaching middle school students about conditional probability and independence – two areas where researchers have identified many student misconceptions (see Table 5.1).

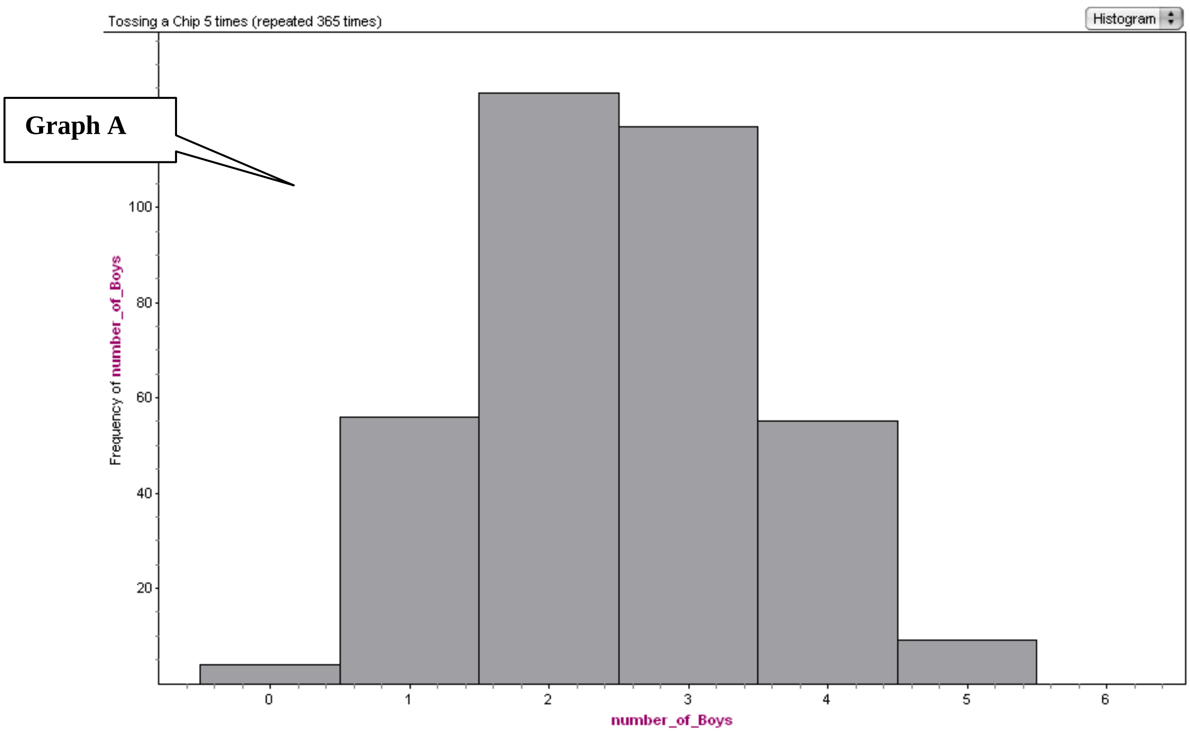
However, before looking at Examples 6.2 – 6.5, it is important to understand that students may struggle with sample size due to over-generalizing their notions about proportionality (Bright, Frierson Jr., Tarr, & Thomas, 2003). Example 6.1 extends the discussion of Example 5.4 by using a simulation to model the situation described.

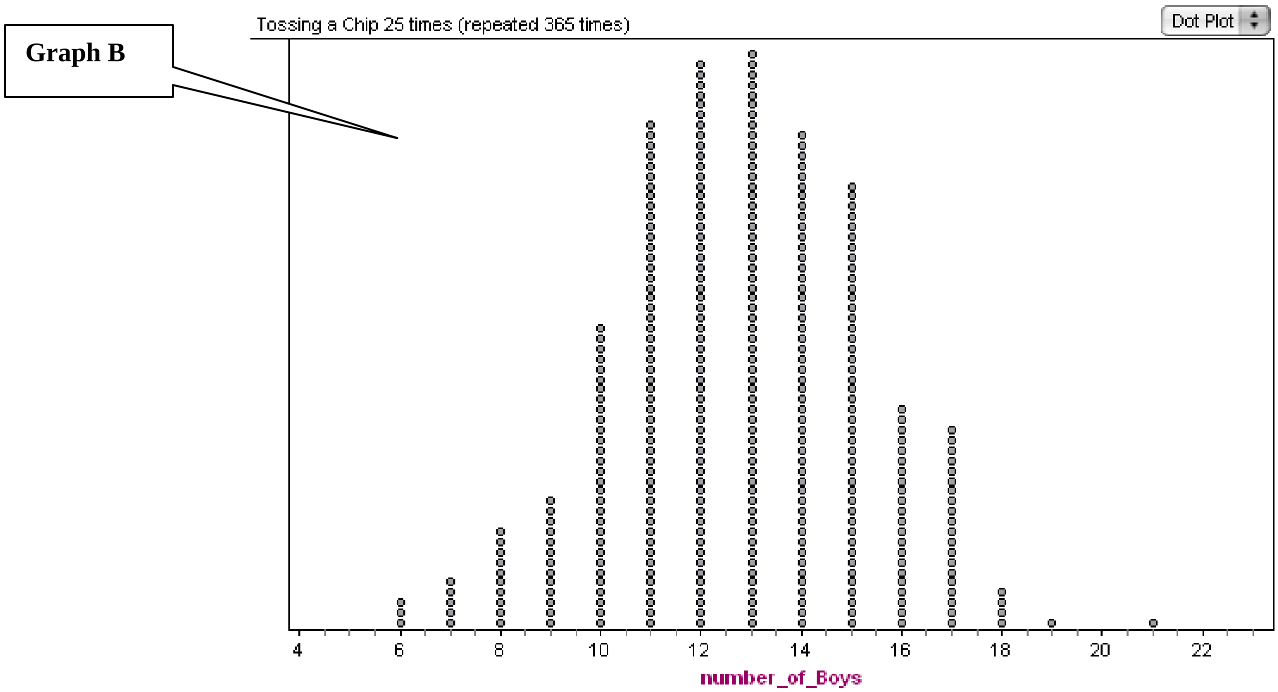
Example 6.1 – Over Generalizing Notions about Proportionality

Take a moment to revisit Essential Question 4 from Section 3. The version below is slightly adapted for illustration purposes – 60% has been changed to 80%.

> **Adapted Essential Question 4** Assume that the chance of having a boy or girl baby is the same. Over the course of a year, in which type of hospital would you expect there to be more days on which at least 80% of the babies born were boys?
>
> A) In a large hospital
> B) In a small hospital
> C) It makes no difference
> (Adapted from Shaughnessy & Bergman, 1993)

Many students will argue that the number of days on which at least 80% of the babies born are boys is independent of the sample size. This may be due to the emphasis on proportionality in the middle school curriculum (Bright, Frierson Jr., Tarr, & Thomas, 2003). However, one would expect there to be more days when at least 80% of the babies born were boys to occur in the small hospital. For example, it is much more likely to obtain at least 80% boys on any given day in a hospital where on average 5 babies were born a day compared to a hospital where 25 babies were born a day (see the note on pg. 35 regarding the difference in sample size). This idea is explored in the activity *Two Hospitals* in Bright, Frierson Jr., Tarr, & Thomas, 2003. The following graphs illustrate this idea and were generated by simulating this activity. Graph A shows the results of tossing a chip with pink on one side (for a girl) and blue on the other side (for a boy) 5 times and recording the number of times the chip comes up blue, then repeating this 365 times. Graph B shows the results of tossing the same chip 25 times and recording the number of times the chip comes up blue, then repeating this 365 times.





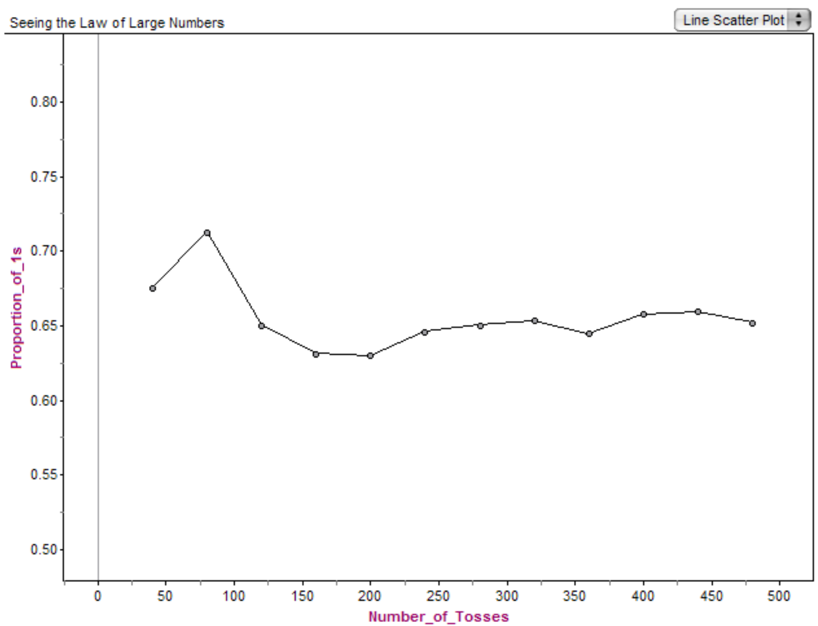
From our simulation, we can see that there were about 65 days in the small hospital when at least 80% of the babies born on a day, over the course of year, were boys (i.e., represented in Graph A by obtaining 4 or 5 on any given day). Whereas, there was only one day in the large hospital, over the course of the year, when at least 80% of the babies born on a day were boys (i.e., indicated in Graph B by any data point greater than or equal to 20).

Exercise 6.1 Explain the Two Hospital simulation and results to a colleague.

The Two Hospitals activity will help middle school students begin to understand that large samples are less likely to yield unusual results and that distributions that are generated from large samples are more likely to represent the parent distribution than distributions generated from small samples. The distributions also give students an idea about what results may be typical. This is the beginning of understanding hypothesis testing. The next example illustrates the Law of Large Numbers.

Example 6.2 – The Law of Large Numbers

Suppose we had a number cube where four of the faces had the numeral 1 on them and the other two faces had the numeral 2 on them. We might ask middle school students to determine the theoretical probability of rolling a 1. While we can use the definition of theoretical probability to calculate the probability of rolling a 1 as $\frac{2}{3}$, it may also be helpful to have students roll the number cube to determine the empirical probability. This type of exercise can help students begin to form the basis of understanding the Law of Large Numbers. That is, students may begin to see that in the short run the results are not too predictable, but in the long run the empirical probability seems to approach $\frac{2}{3}$. In a classroom setting, one can ask each pair of students to roll the number cube 40 times and record the number of times the cube turns up a 1. Then, these results can be aggregated for the entire class. For example, if a class has 24 students then each of the 12 pairs will generate 40 rolls, and, hence, in the aggregate there will be 480 rolls. For each added pair (40 more rolls) we could produce a graph, similar to the one below, showing the number of rolls and the relative frequency of obtaining a 1. Students will then begin to see the Law of Large Numbers.

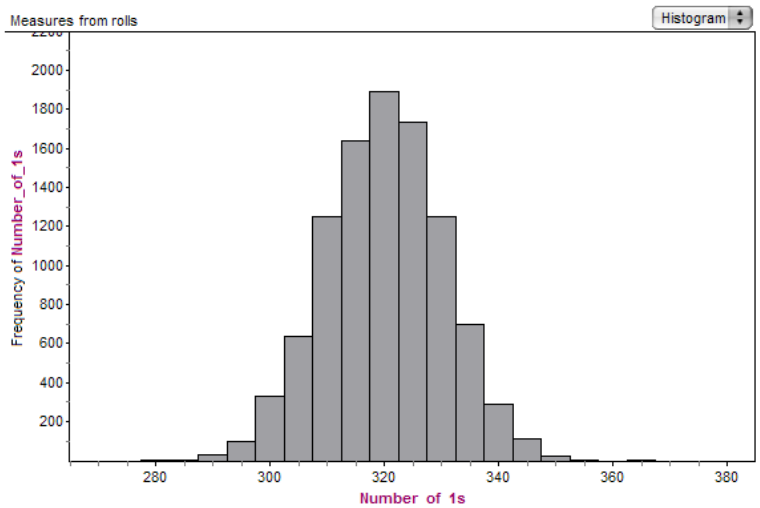


In addition, we typically ask questions like the following:

> For the number cube described above, what is the most likely number of 1s you would expect to roll in 480 rolls?

Take a moment to think about this question.

While we usually approach this problem from a theoretical standpoint (i.e., the probability of rolling a 1 is $\frac{2}{3}$, therefore, the most likely number of 1s you would expect in 480 rolls is $\frac{2}{3} \cdot 480 = 320$), there is a nice opportunity here to talk about probability distributions and to connect this procedural knowledge to conceptual understanding by using data to generate the probability distribution. That is, we can conduct the experiment (rolling this number cube 480 times) many times and record the number of 1s obtained each time we roll the number cube 480 times. Since repeating the experiment of rolling a number cube 480 times isn't practical, we turn to technology. Using simulation software, we can easily ask a program to repeat this experiment many times, say 10,000 times, and generate a probability distribution like the one below. Notice that the number of 1s clump around 320.

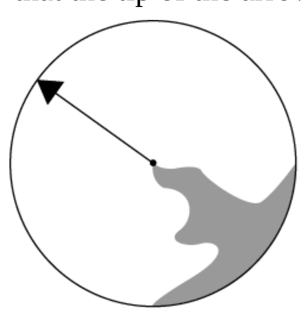


In addition to producing a histogram, dynamic software will allow you to produce a dot plot and watch the plot change as new cases are added. So, students can begin to see the distribution take shape.

Additionally, students can see that other numbers of 1s are possible and which of those other numbers are somewhat reasonable (i.e., which number of 1s is somewhat near the middle of the distribution). This basic understanding will contribute to students' understandings of hypothesis testing.

The above distribution is based on empirical probability. We could generate the distribution by calculating theoretical probabilities. That is, we could use counting techniques to count the number of ways that we could obtain 320 1s.

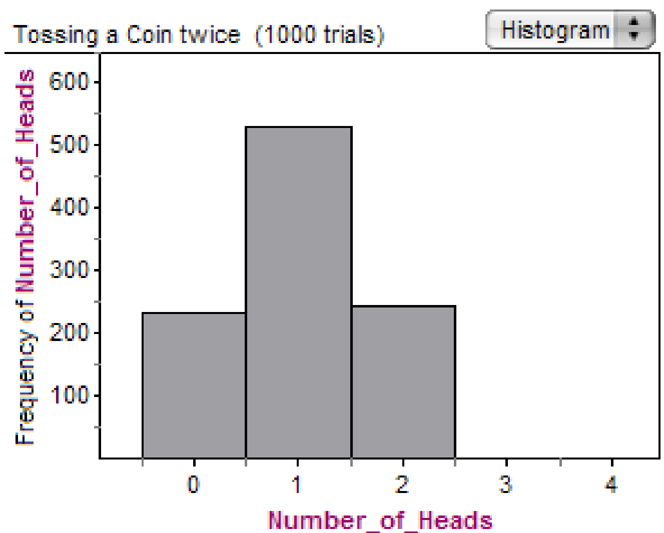
Exercise 6.2 Go back and revisit Essential Question 2 from Section 3.

> **Essential Question 2** The arrow on the spinner below is spun once. What is the approximate probability that the tip of the arrow will land in the shaded region?
>
> 
>
> (Adapted from Shaughnessy et al., 2004)

Explain the connection of this question to the Law of Large Numbers.

Example 6.3 – Emphasizing Sample Space

Suppose you were to ask students to toss a fair coin twice to determine the number of possible heads you could obtain. Students might initially determine that you can obtain 0 heads, 1 head, or 2 heads, but incorrectly believe that these outcomes are equally likely. That is, if we were to toss a fair coin twice there would be a one-third chance of obtaining each of the above outcomes. Having students perform the experiment can help them determine that they can obtain 1 head in two different ways (i.e., HT or TH), and that obtaining 1 head is twice as likely as obtaining 0 heads, and twice as likely as obtaining 2 heads. Performing the experiment and recording the results will lead to an experimental distribution that resembles the distribution shown below.



Example 6.4 – Connecting Probability and Statistics

Many of the examples that we have already studied should help you understand the connection between probability and statistics. With that in mind, take a moment to revisit Essential Question 12 from Section 3.

> **Essential Question 12** How does probability connect to statistics? Explain.

The GAISE framework contains a nice description of how probability relates to statistics. The framework describes probability as a tool for statistics and illustrates the difference between probability as a mathematical model and probability as a tool in statistics with the following two examples (Franklin, Kader, Mewborn, Moreno, Peck, Perry, & Scheaffer, 2005):

- Probability as a mathematical model: Suppose you toss a fair coin five times, how many heads will we get?

- Probability as a tool for statistics: Suppose you picked up a coin. Is the coin fair?

The second question can be answered by conducting an experiment – toss the coin and find out what happens by recording the results. Then, compare the results to what we know about results for a fair coin. That is, do the results seem typical of those from a fair coin? Whereas, the first problem asserts that the coin is fair and deals with determining the probability of the various numbers of heads.

Exercise 6.3 To see some examples of activities that focus on connecting probability and statistics, examine Chapter 4 of Bright, Frierson Jr., Tarr, & Thomas, 2003.

Example 6.5 – Introduction through Data

When introducing probability concepts through data, it is important to keep in mind that many situations involving data can have solutions that seem quite complex to students. It may be best to begin by providing students with data and asking them to compute probabilities that can easily be determined from the data. However, it is also important to challenge students' understandings of the concepts. Essential Question 5 from Section 3 illustrates a situation where computing a probability based upon data can be difficult partially due to the use of particular heuristics.

Take a moment to revisit Essential Question 5 from Section 3.

> **Essential Question 5** There are two cab companies that operate in the city, a Blue Cab company and a Green Cab company. It is known that 85% of the cabs in the city are Green and 15% are Blue. A cab was involved in a hit-and-run accident at night. A witness at the scene identified the cab involved in the accident as a Blue cab. This witness was tested under similar visibility conditions, and made correct color identifications in 80% of the trial instances. Given that the witness identified the cab as blue, what is the probability that the cab involved in the accident was a Blue cab rather than a Green one? Explain your reasoning.
>
> (Adapted from Shaughnessy & Bergman, 1993)

As you explored in Section 5 Exercise 5.4, the representativeness heuristic may be used to judge the likelihood of an event happening where the base-rate information is neglected. Results of Essential Question 5 indicate that people tend to neglect that only 15% of the cabs in the city are blue; they tend to rely on the witness account since the witness is accurate 80% of the time. That is, people feel that the single incident of the accident occurring should be representative of the 80% reliability of the witness (Shaughnessy & Bergman, 1993). Using the data to create a contingency table can help to examine the results. The following table is based on a sample of 100 cabs where the witness is 80% accurate and 85% of the cabs in the city are green.

	Cab Color		
Color reported by Witness (80% accurate)		Blue	Green
	Blue	12	17
	Green	3	68

From this table, given that the witness reported the cab involved in the accident as being blue, the probability that the cab was blue is 12/29 or about 41%.

Exercise 6.4 Create some examples appropriate for your class that introduce probability concepts through data. Make certain that some of the examples are designed to purposely assess whether or not certain misconceptions may be present.

Example 6.6 – Informal Hypothesis Testing & Large-scale Experiments: Spinning Pennies

Hypothesis testing is just as its name implies – about testing a hypothesis. That is, we develop a hypothesis about something and then conduct experiments and collect data to test that hypothesis. The fundamental question lies in determining whether or not we should reject or not reject our hypothesis. That is, if we assume that our hypothesis is correct and then we gather data, we need to understand what the likelihood is that we could have found these data purely due to chance. If our outcome is a reasonably likely outcome for the given hypothesis, then we would say that the result is **not** statistically significant. We will illustrate these ideas and the fact that the spread of the data depends on the size of the sample (i.e., results are likely to vary more in small samples compared to large samples) by examining Essential Question 10 from Section 3.

> **Essential Question 10**
>
> Suppose you were to spin a penny 40 times. How many times would you expect the penny to land on heads?

If we were to spin a penny 40 times, it would be reasonable to assume that 50% of the time the penny should land on heads. That is, we might hypothesize that obtaining heads or tails when spinning a penny is equally likely. Now that we have established a hypothesis, we wish to test the hypothesis.

Let's begin by spinning a penny 10 times and recording the number of heads that we get. Suppose we obtain 4 heads, or that 40% of the time the penny landed on heads. Does this give us enough information to reject our original assumption that we should obtain heads 50% of the time? To answer this question, we need to realize that obtaining 4 heads in 10 spins is certainly a possibility. The question now becomes, how likely is this result? If the result isn't a very likely outcome, we would say that the result is statistically significant. There are a couple of ways that we might go about determining how likely this result is. The first is by using counting techniques. That is, we could determine the theoretical probability of obtaining exactly 4 heads in 10 spins of a penny if our proposed hypothesis is true (i.e., that we should obtain heads 50% of the time). To determine this, we need to answer two questions:

1) How many possible equally likely outcomes are there when we spin a penny 10 times (again assuming that obtaining heads is equally likely as obtaining tails)?
2) How many of these outcomes have exactly 4 heads?

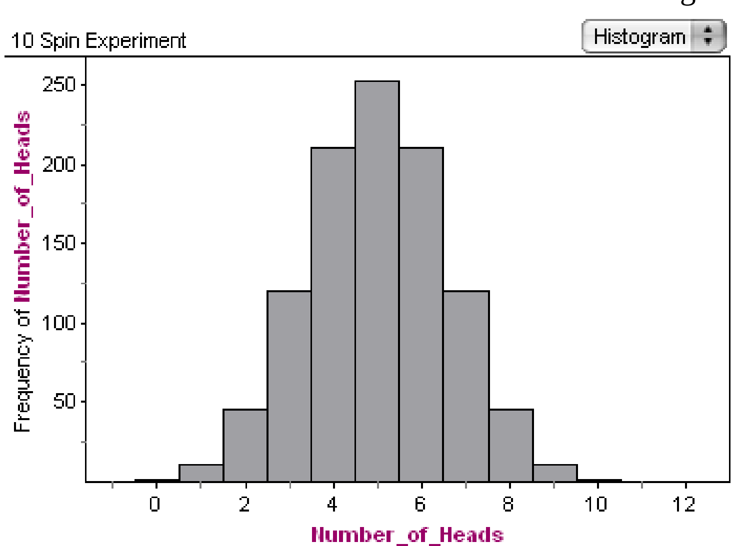
Since each spin can land on either heads or tails, there are 2 possible outcomes for each of the 10 spins, or $2^{10} = 1024$ total outcomes (using the multiplication principle). To find the number of outcomes that have exactly 4 heads, if we think of ten slots representing each of the 10 spins, we note that any 4 of these could be heads. We have 10 choices for which slot could be the first slot with an H, then 9 choices for the second slot with an H, 8 choices for the third slot with an H, and 7 choices for the fourth slot with an H. However, the order of these H's among these four slots isn't important, so we need to divide by 4!. Hence, there are $\frac{10 \cdot 9 \cdot 8 \cdot 7}{4!} = 210$ outcomes with exactly 4 heads. So, the probability of obtaining 4 heads in 10 spins is $\frac{210}{1024} \approx 20.5\%$. The following table shows the different number of heads possible and the number of outcomes containing exactly the given number of heads.

Table 6.3 – 10 Spin Experiment

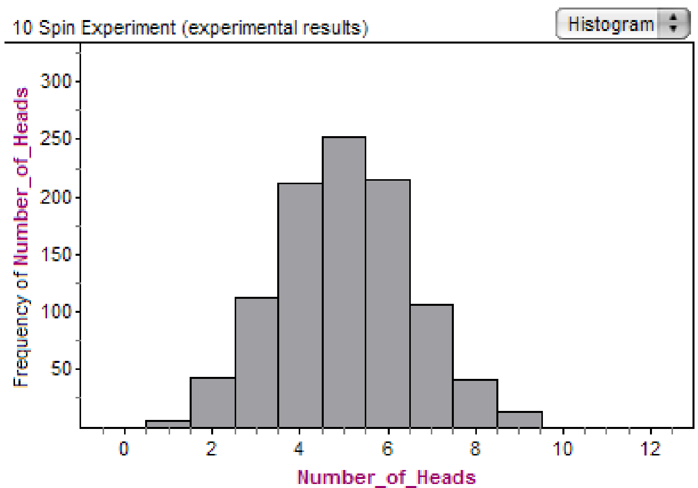
Number of Heads	0	1	2	3	4	5	6	7	8	9	10	**Total**
Frequency of Number of Heads	1	10	45	120	210	252	210	120	45	10	1	1024

Exercise 6.5 Verify the information in the above table.

We can use the information in Table 6.3 to construct the following histogram.

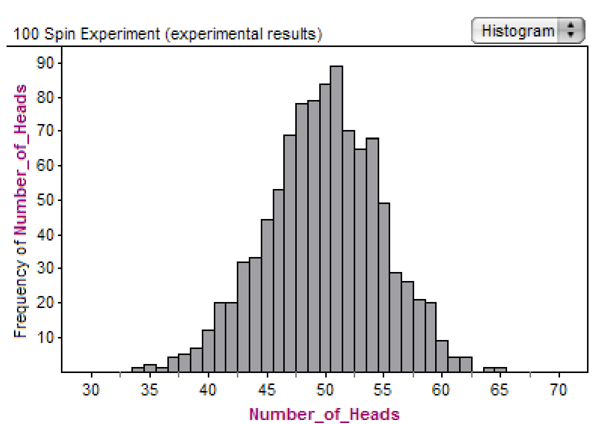


Alternatively, we could simulate the experiment and create a histogram from experimental data. Here is the result of simulating spinning the penny 10 times and repeating this experiment 1000 times.



In either case, we can see that obtaining 4 heads in 10 spins isn't such an unusual result. Thus, we do not have sufficient evidence to reject our hypothesis that the coin should land on heads 50% of the time. And, we would say that our result of obtaining 4 heads in 10 spins isn't statistically significant. In fact, we can see from the histograms that we would expect that most of the time we would get 4, 5, or 6 heads when spinning a penny if it were equally likely to land on heads as tails.

Now, let's suppose we spin our penny 100 times and we obtain 40 heads. This result may encourage us to re-think our original hypothesis. That is, this result may provide evidence that our original hypothesis is false. To help determine this, we again have to ask how likely is it that we would obtain 40 heads in 100 spins if obtaining a head or a tail was equally likely. We could generate a table similar to Table 6.3 for 100 spins and generate the related histogram. However, we will choose to simulate the experiment and generate a histogram from the experimental data. The following is the result of spinning the penny 100 times and recording the number of heads when this experiment is repeated 1000 times.



From our experimental results, we can see that we should expect in 100 spins to obtain a number of heads that clusters around 50, if it is truly the case that our penny is equally likely to land on heads as tails. Based on our results, obtaining 40 heads in 100 spins occurred 12 out of 1000 times or 1.2% of the time. So, obtaining 40 heads in 100 spins looks to be a rare case if there is really a 50% chance of obtaining heads. Thus, we are beginning to think that we should reject our original hypothesis. And, we are beginning to see that obtaining heads 40% of the time is much rarer in 100 spins than in 10 spins – as the sample size increases the spread decreases. If we were to repeat the experiment for 1000 spins, we would see that it would be even rarer to obtain heads 40% of the time. Therefore, we have sufficient evidence to reject our original hypothesis. Thus, when spinning a penny, obtaining heads or tails is not equally likely. In fact, we see that it may be the case that we should expect to obtain heads about 40% of the time. When students study statistics in high school they will learn to quantify the level at which their results are statistically significant and learn about confidence intervals.

For example, you may hear that an election poll reports that a certain candidate has received 53% of the vote with a ± 4 margin of error. This is not an error in the result of the gathering of the data, but rather a statement about the inference of the results from the sample to the population. That is, the poll certainly didn't ask all voters how they were going to vote – they simply took a sample of all voters. And, in that sample, 53% of the voters stated that they would vote for a certain candidate. However, on Election Day, this candidate could receive any number of votes – perhaps 55% or 45%. In either case, there is some chance that if the candidate received, say, 45% of the vote, that we happened to obtain, by luck, a sample of people where 53% of those people voted for the candidate. So, the statement that the candidate received 53% of the vote with a ± 4 margin of error means that on election day it is probably the case that the candidate will receive between 49% and 57% of the vote. In reality, we quantify what we mean by "probably" and might state that we are 95% confident that the candidate will receive between 49% and 57% of the votes. What this really means is the following: If the actual percent of votes for this candidate lies outside of the 49% to 57% range, then there would be less than a 5% chance of randomly obtaining 53% as a sample outcome (if we were to take 100 random samples from the population and compute the resulting 100 95%-confidence intervals, we would expect 95 of them to capture the true population proportion).

Building these ideas in an informal way through experiments and simulations will help students begin to understand statistical inference.

We end this section by discussing specific strategies to address teaching conditionals and independence, as shown in Table 6.4. Take a moment to revisit Table 5.1 to review students' conceptions about conditionals.

Table 6.4 – Strategies for Teaching Conditional Probability and Independence (Adapted from Tarr & Lannin, 2005)

Strategy	Explanation and Examples
Design of Problem Tasks	Instructional plans that focus on the ideas of conditional probability and independence should be centered on familiar contexts and should include key questions that promote small-group and whole-class discussions. These activities should challenge students' understandings of concepts in probability and require them to resolve possible misconceptions or conflicts; for example, questions similar to Exercise 4.9. Students could examine this situation in a familiar context, similar to those described in Example 5.12 or through an activity (e.g., rolling a six-sided number cube and determining the probability that a 3 is displayed given that the result is odd versus the probability of obtaining an odd number given that the result is a 3).

Table 6.4 continues on the next page.

Table 6.4 Continued…

Relating Sample Space and Probability of an Event to Conditional Probability and Independence	The ability to connect "sample space" and "probability of an event" is a key factor in students' understandings of conditional probability. Students need to be able to realize that the probability of all events change in non-replacement situations. For example, suppose a candy is selected from a jar that contains 4 grape candies, 5 lemon candies, and 6 orange candies. Once the candy is selected it is eaten. If the first candy selected was a grape candy, on the next selection, has the chance of obtaining a grape candy changed? You might ask students to focus on the total number of candies before and after the grape candy is selected. Then, teachers might ask students what would happen if the grape candy was replaced after being selected (i.e., how would the probabilities of each event change?). To contrast this last example, you might have students work with a task where the events are independent. For example, a fair coin is flipped repeatedly. On the first flip a head (H) was obtained, and on the second flip a tail (T) was obtained. Which outcome is most likely for the next flip: H, T, or these outcomes are equally likely? See Example 4.6. Focusing on the sample space after each trial can help students to see that the sample space doesn't change in replacement situations and therefore the probability of events remains unchanged.
Focusing on Conditional Probability and Independence Simultaneously	Researchers suggest introducing independence as a special case of conditional probability. See the discussion in Section 4. This approach allows students to focus on whether or not the probability of events change or stay the same and promote focusing on the sample space (i.e., sample space is restored in replacement situations and changes in non-replacement situations).

Table 6.4 Continued…

Using Simulations	There is evidence that simulations foster students' understanding of some probability concepts, but little evidence that they foster students' understandings of conditional probability. However, simulations may help students with their conceptions about independence. In particular, they may challenge students' use of the representativeness heuristic. Researchers caution that small samples of situation data may serve to validate students' misconceptions. Hence, teachers should focus on having students make predictions over the long term.

Exercise 6.6 To gain a greater understanding of the above strategies and to review a framework for assessing students' thinking in conditional probability and independence, read chapters 8 and 9 of Jones, 2005.

Exercise 6.7 To see an activity that examines conditional probability and independence explore http://www.shodor.org/interactivate/lessons/ConditionalProb/.

Now that we have examined students' understandings of concepts on counting techniques and probability, along with some of the most common misconceptions and also teaching strategies, you will spend some time looking through the released items from the NECAP assessment and answer some questions about them that help to place the released items within the context of this unit of study.

Section 7: Examining NECAP Released Items and Student Work

This section contains exercises focused on items released from the New England Common Assessment Program, along with the sample student work (for constructed response items). Each released item is mapped to a primary Grade-Level Expectation as indicated by the codes in the released item support materials document. These Grade-Level Expectations can be located in Section 2. These exercises will allow you to explore how students approach some problems that connect to counting techniques and probability, and provide the opportunity for you to make connections between these items and the research highlighted in this unit of study.

Exercise 7.1 Locate all of the released NECAP items and practice test items, along with the released item support materials that deal with the GLEs highlighted in Section 2. These items can be located at the following links.

NECAP Practice Test Items

http://www.ed.state.nh.us/education/doe/organization/curriculum/NECAP/PracticeTest.htm

2005 and 2006 Released NECAP Items

http://www.ed.state.nh.us/education/doe/organization/curriculum/NECAP/Released%20Items/ReleasedItems.htm

Exercise 7.2 For each of the multiple choice questions located in Exercise 7.1, determine how students might obtain each of the answer options and how these options connect to the research highlighted in this unit of study.

Exercise 7.3 Read through the student work that is released for each of the constructed response items found in Exercise 7.1 Determine, based on the rubric provided, how the various responses were scored. Then determine how the students' responses connect to the research highlighted in this section.

Exercise 7.4 Select some of the available student responses that illustrate some of the misconceptions highlighted in this unit of study. Then, discuss some various teaching strategies that could be used to address these misconceptions.

Section 8: Summary

This section contains five exercises focused on examining your current curricular materials and creating an instructional unit on counting techniques and probability. This section is best completed with the collaboration of colleagues and requires extensive planning. The goal is to integrate the research presented in this unit of study into your lesson plans and instruction.

Exercise 8.1 Examine your instructional materials and district curriculum and determine if they provide opportunities for students to test personal conjectures, engage in large-scale experiments. Do they emphasize a problem-solving approach that engages students in tasks involving counting techniques or probability and statistics? Do they contain tasks that introduce probability through data?

Exercise 8.2 Examine your instructional materials and district curriculum and determine if they emphasize a connection between probability and statistics. While doing this, it is recommended that you spend time reading the GAISE framework (see the link on the next page). This framework will help you understand the four components of the statistical process (formulating questions, collecting data, analyzing the data, and interpreting the results), along with the nature of variability in each of these components (anticipating variability when formulating questions, designing data collection methods that acknowledge variability, using distributions to analyze the data, and allowing for variability when interpreting results). The framework also contains a nice selection of rich activities adaptable for the classroom.

Exercise 8.3 Examine your instructional materials and district curriculum and determine if they emphasize a connection between probability and proportionality.

Exercise 8.4 Examine your instructional materials and district curriculum and determine if they provide ample opportunities to challenge students' understandings of probability concepts and assess whether or not students exhibit any of the misconceptions outlined in Table 5.1.

Exercise 8.5 Create a curricular sequence/unit of study for the topic of counting techniques and probability. The sequence should consider all three principles of how students learn. The following table can serve as a model for your work.

Concept	Description of How Concept is Introduced	Target Depth of Knowledge Levels*	Description of Activities

*You may want to review Appendix A for information on Depth of Knowledge.

This unit of study was meant to introduce you to research around counting techniques and probability that can have a direct impact on classroom instruction. It was intended to be used as a supplement to current curricula materials and provide examples that challenge students' understandings of these concepts.

If you are looking to build upon the ideas presented in this unit of study and want to increase your depth of understanding of this vast topic, you are encouraged to explore the references listed in Section 10. In addition to the resources in Section 10 and those contained in exercises within the sections of this unit of study, you might want to explore the following sites (as of October 2007) for standard-based activities connected to probability and statistics:

National Council of Teachers of Mathematics Illuminations Web Site:
http://illuminations.nctm.org/

PBS Teachers:
http://www.pbs.org/teachers/

Annenberg Media Learner.org:
http://www.learner.org/

Balanced Assessment:
http://balancedassessment.concord.org/

Guidelines for Assessment and Instruction in Statistical Education (GAISE):
http://www.amstat.org/education/gaise/

Resources for Teaching and Learning about Probability and Statistics – Eric Digest:
http://www.ericdigests.org/2000-2/resources.htm

Federal resources for educational excellence:
http://free.ed.gov/

National Library of Virtual Manipulatives:
http://nlvm.usu.edu/en/nav/vlibrary.html

Section 9: Answers to Exercises

Section 1: Purpose and Design

Exercise 1.1: Answers will vary.

Section 2: Connecting to the Grade-Level Expectations

Exercise 2.1: Primary GLEs are given on pp. 5–6 of Section 2.

Exercise 2.2: See the Curriculum Focal Points document from NCTM to find these.

Section 3: Essential Questions

Essential Question 1: is as likely as; is more likely than (this item is discussed in Example 5.3)

Essential Question 2: c (this item is discussed in Example 4.3)

Essential Question 3: Alisa is not correct. The probability of both arrows landing on the gray section is ¼ since there are four equally likely outcomes of which only one is both arrows landing on the shaded gray section (left on gray, right on gray; left on gray, right on white; left on white, right on gray; left on white, right on white). See Example 5.11.

Essential Question 4: B (this item is discussed in Examples 5.4 and 6.1 – see the note regarding the magnitude of the difference in the number of babies born in Example 5.4)

Essential Question 5: about 41% (this item is discussed in Example 6.5)

Essential Question 6: Since theoretically each of these outcomes is equally likely, there should be about 72 families where the exact order of the births was BGBBBB.

Essential Question 7: Both events are equally likely since the outcome of tossing the coin the second time doesn't depend upon the outcome of the first toss, and the outcome of tossing the coin the third time doesn't depend upon the outcome of either of the previous two tosses.

Essential Question 8: d (this item is discussed in Example 5.9)

Essential Question 9: part a) 367 considering February 29 (see Example 4.5 for an explanation); part b) 23 (see Example 4.5 for an explanation); 253 (see Example 4.5 for an explanation)

Essential Question 10: the answer depends upon the date on the pennies, but it may be close to 40% and you might determine whether or not your prediction seems accurate by performing a hypothesis test (see Example 6.6). You might want to think about why the expectation isn't 50% heads.

Essential Question 11: You might look at the randomness of the sequence including the runs of a particular outcome (see Example 4.6).

Essential Question 12: This exercise is integrated throughout the unit of study; however, in particular, see Example 6.4.

Section 4

Exercise 4.1: The sample space can be expressed as {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}, which represents the possible sums; however, these sums are not equally likely. The 36 equally likely outcomes can be seen in the following table. Note, even though it appears that some outcomes in the chart repeat, we are thinking of each of these outcomes as being unique based upon how the outcome was generated. For example, the 3 generated from a 2 on the first die and a 1 on the second die is not the same as the 3 generated from the 1 on the first die and a 2 on the second die. It may be helpful to color the die – for instance, the first die could be white and the second green. Then, you could make a chart that shows the numbers on each die for a particular outcome. Visually, this helps distinguish between the outcomes.

Number on second number cube

Number on first number cube

+	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Exercise 4.2: 720; 8,000,000 (assuming the first position can't be a 0 or 1 and ignoring some other constraints); 12; 8,145,060; 120; 210; 720 (note: this assumes that a rotation of a particular arrangement doesn't count as a different arrangement)

Exercise 4.3: Since $2160 = 2^4 \cdot 3^3 \cdot 5$, there are 5 × 4 × 2 = 40 positive divisors (to determine the total number of factors, note that you select anywhere from 0 to 4 factors of 2, 0 to 3 factors of 3, and 0 or 1 factor of 5).

Exercise 4.4: See the illuminations web site.

Exercise 4.5: Answers will vary. Sample answer: assuming that the most hairs anyone in the world could have per square inch on his or her body was 20,000 (try counting how may hairs you have on your scalp in a quarter inch by quarter inch area to convince yourself that this is a safe assumption) and the largest person on earth has no more than 30,000 square inches of skin (again, convince yourself that this is a safe assumption), then no person on the earth can have more than 600,000,000 hairs on his or her body, but the population of earth is over 6 billion. Hence, by the Pigeonhole Principle, there must be two non-bald people on the earth with the same number of hairs on their bodies.

Exercise 4.6: 6 ways; 27 (note: this assumes repetition is allowed – the context implies repetition); 4 ways

Exercise 4.7: See the activity in the *Navigating through Probability in Grades 6-8* materials.

Exercise 4.8: Simulations will vary. Sample simulation: Have one person roll a six-sided number cube to decide which door to hide the prize behind (rolling a 1 or 2 means hide the prize behind door 1, rolling a 3 or 4 means hide the prize behind door 2, and rolling a 5 or 6 means hide the prize behind door 3). Then, have another person roll the number cube to determine the initial pick. Finally, explore each strategy (i.e., switching vs. sticking) and compare how likely you are to win by implementing each strategy (you might also want to explore a mixed strategy).

Exercise 4.9: No, in general these probabilities are not equal. For example, consider rolling a six-sided number cube and determining the probability that a 3 is displayed given that the result is odd versus the probability of obtaining an odd number given that the result is a 3. The first probability is 1/3 while the second probability is 1.

Exercise 4.10: 3/5

Exercise 4.11: Explanations will vary, but note that the table involves calculating a ratio, the tree diagram and area model involve reading the probability using the correct branch or area.

Exercise 4.12: 31/64 or about 0.484; explanations will vary depending on model.

Exercise 4.13: 217/640 or about 0.339; explanations will vary depending on model.

<u>Section 5</u>

Exercise 5.1: Answers will vary. Sample answer: This particular expectation focuses on the relationship between the part and the whole when the wholes may vary in magnitude. Students' conceptions about proportionality can sometimes affect their reasoning about situations involving probability (e.g., see Essential Question 4 from Section 3). Understanding, for example, that 40% of 100 is not the same as 40% of 1000 is important when reasoning about situations involving probability.

Exercise 5.2: Answers will vary. Sample answer: Students who struggle with the ratio concept may incorrectly identify Spinner A as having the greater probability of landing on a section numbered 1 – reasoning either that Spinner A has more sections (3) that are numbered one or that the probability for Spinner A is 3/8 compared to 1/5 for Spinner B.

Exercise 5.3: A person exhibiting the representativeness heuristic might argue that the result of the next toss should be a T in order to compensate for the run of heads (negative recency) or perhaps pick that the next toss should be an H since the prior outcomes are all H (positive recency). Revisit the discussion in Example 4.6.

Exercise 5.4: Representativeness often affects students' reasoning on this problem. They tend to neglect the base-rate information (only 15% of the cabs in the city are blue) and rely too heavily on the eye witness account, since the witness is accurate 80% of the time. See the discussion in Example 6.5.

Exercise 5.5: Questions will vary.

Exercise 5.6: Answers will vary. See the task described in Exercise 5.7.

Exercise 5.7: See the discussion in the article.

<u>Section 6</u>

Exercise 6.1: Explanations will vary. Review the explanation in Example 6.1.

Exercise 6.2: This question helps students understand probability as a long-run relative frequency of an outcome. That is, as the number of spins increases the relative frequency of an outcome comes closer to the true probability.

Exercise 6.3: See Chapter 4 of *Navigating through Probability in Grades 6-8*.

Exercise 6.4: Examples will vary.

Exercise 6.5: Verifications will vary, but may follow the explanation given for 4 heads in Example 6.6. For example, $\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5!}$ is an expression that represents how to find the total number of outcomes with exactly 5 heads.

Exercise 6.6: See Chapters 8 and 9 of Jones, 2005.

Exercise 6.7: See http://www.shodor.org/interactivate/lessons/ConditionalProb/.

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Appendix A – Implied Cognitive Demand and Depth of Knowledge

A fundamental criterion used to develop the NECAP GLEs and GSEs is that the expectations should explicitly indicate cognitive demand (how content interacts with process) and that there should be a mix of cognitive demand levels at all grades. That is, one should not assume that students at lower grades do less cognitively demanding work. The cognitive demand or depth of knowledge required by an expectation or an assessment item is related to the number and strength of connections of concepts and procedures that a student needs to make to produce a response, including the level of reasoning required along with self-monitoring. Furthermore, there are additional factors that influence cognitive demand including contextual requirements, language, the number and variety of representations, requirements for generalizations to new situations, and the opportunity to learn.

It is important to note that depth of knowledge is not synonymous with difficulty. As an example, solving a multi-step linear equation with variables on both sides may be a difficult task for middle school students; however, the task can be solved by applying a standard procedure making the task of low complexity.

The NECAP states believe that expectations and assessment should be aligned in terms of their cognitive complexity. That is, the cognitive complexities of the assessment items should match that of the standards (what students are expected to know and be able to do). To ensure this alignment, the NECAP states have adopted Norman L. Webb's (senior researcher with the Wisconsin Center for Educational Research) Depth of Knowledge classification system. Norman Webb's system is based on four levels of classification. The full descriptions of each level are given on pages 4 and 5. The levels can be summarized as follows.

Level 1	Recall
Level 2	Skill/Concept
Level 3	Strategic Thinking
Level 4	Extended Thinking

The NECAP states, together with a committee of educators, analyzed the GLEs and GSEs for their implied cognitive demand. That is, all aspects of each expectation were analyzed and the implied cognitive demand levels were recorded. One of the charges of the NECAP test item review committees is to ensure that assessment items align not only with the expectations but also with their implied cognitive demands. The range of cognitive demands for each GLE and GSE is summarized in Table 1 on page 2. It should be noted that the highest level listed for each GLE and GSE should be thought of as a "ceiling" not a "target". That is, the goal is to write items which cover the range of the levels indicated and not just the highest level. If one assesses only at the "target" level, all GLEs with a level 3 (for example) as their highest cognitive demand would only be assessed at level 3. This would potentially have two negative impacts on the assessment: 1) The assessment as a whole would be too difficult; and 2) important information about student learning along the achievement continuum would be lost. To the extent possible, GLEs and GSEs should be assessed at the "ceiling" and at least one level below the "ceiling" in order to provide additional diagnostic information to educators. Furthermore, Table 2 shows an example of an expectation and how the different aspects of the expectation interact with Table 1.

Table 1

	Depth of Knowledge Levels for NECAP Assessment						
	2	**3**	**4**	**5**	**6**	**7**	**10**
M(N&O)–X–1	1, 2	1, 2	1, 2	1, 2	1, 2	1, 2	
M(N&O)–X–2	1	2	2	2	2	2	1, 2, 3
M(N&O)–X–3	1, 2	2	2	2,3	2,3		
M(N&O)–X–4		1, 2, 3	1, 2, 3	1, 2, 3	1, 2, 3	1, 2, 3	1, 2, 3
M(N&O)–X–5	1, 2						
M(G&M)–X–1	1, 2, 3	1, 2	1, 2	1, 2	1, 2		
M(G&M)X–2						1, 2	1, 2, 3
M(G&M)–X–3			1, 2	1, 2	1, 2		
M(G&M)–X–4				1, 2		1, 2	2, 3
M(G&M)–X–5			1, 2		1, 2	1, 2, 3	1, 2, 3
M(G&M)–X–6	1, 2	1, 2	1, 2	1, 2	1, 2, 3	1, 2, 3	1, 2, 3
M(G&M)–X–7	This GLE will NOT be directly assessed but embedded in problems in other content strands.						1, 2
M(G&M)–X–8							
M(G&M)–X–9							2, 3
M(F&A)–X–1	2	2	2	2	2, 3	2, 3	2, 3
M(F&A)–X–2					1, 2	1, 2, 3	1, 2, 3
M(F&A)–X–3			1	1	1, 2	1, 2	1, 2
M(F&A)–X–4	1	1, 2	1, 2	1, 2	1, 2	1, 2	
M(DSP)–X–1	1, 2, 3	1, 2, 3	1, 2, 3	1, 2, 3	1, 2, 3	1, 2, 3	2, 3
M(DSP)–X–2	2, 3	2, 3	2, 3	2, 3	2, 3	2, 3	2, 3
M(DSP)–X–3		1, 2		1, 2		2, 3	1, 2, 3
M(DSP)–X–4	2		2, 3		2, 3		1, 2, 3
M(DSP)–X–5		1, 2	1, 2	1, 2	1, 2, 3	1, 2, 3	1, 2, 3

Black cells indicate GLEs or GSEs that are not assessed on NECAP at the given level.

Sample Mathematics GLE* for End of Grade 6	Potential DoK Levels	DoK Ceiling	Aspects of GLE at different levels**
M(F&A)–6–1 **Identifies and extends to specific cases a variety of patterns** (linear and nonlinear) represented in models, tables, sequences, graphs, or in problem situations; or writes a rule in words or symbols for finding specific cases of a linear relationship; or writes a rule in words or[sc] symbols for finding specific cases of a nonlinear relationship; and writes an expression or[sc] equation using words or[sc] symbols to express the **generalization** of a linear relationship (e.g., twice the term number plus 1 or[sc] $2n + 1$).	**2, 3**	**3**	Level 2 **Extends a pattern to a specific case** **Level 3** Generalizes a pattern

Table 2

*GLE NOTES: Underlining in the GLE indicates that this concept or skill is "new" to grade 6 for assessment purposes. The superscript "sc" indicates that students have a choice in how they complete the task (e.g., students can use words **or** symbols to express the rule).

**Recall, one must also consider other factors when making decisions on Depth of Knowledge levels such as contextual requirements, language, the number and variety of representations, requirements for generalizations to new situations, and the opportunity to learn.

Depth of Knowledge Descriptors for Mathematics
Norman L. Webb
March 28, 2002

Mathematics Depth of Knowledge Levels

Level 1 (Recall) includes the recall of information such as a fact, definition, term, or a simple procedure, as well as performing a simple algorithm or applying a formula. That is, in mathematics a one-step, well-defined, and straight algorithmic procedure should be included at this lowest level. Other key words that signify a Level 1 include “identify,” “recall,” “recognize,” “use,” and “measure.” Verbs such as “describe” and “explain” could be classified at different levels depending on what is to be described and explained.

Level 2 (Skill/Concept) includes the engagement of some mental processing beyond a habitual response. A Level 2 assessment item requires students to make some decisions as to how to approach the problem or activity, whereas Level 1 requires students to demonstrate a rote response, perform a well-known algorithm, follow a set procedure (like a recipe), or perform a clearly defined series of steps. Keywords that generally distinguish a Level 2 item include “classify,” “organize,” ”estimate,” “make observations,” “collect and display data,” and “compare data.” These actions imply more than one step. For example, to compare data requires first identifying characteristics of the objects or phenomenon and then grouping or ordering the objects. Some action verbs, such as “explain,” “describe,” or “interpret” could be classified at different levels depending on the object of the action. For example, if an item required students to explain how light affects mass by indicating there is a relationship between light and heat, this is considered a Level 2. Interpreting information from a simple graph, requiring reading information from the graph, also is a Level 2. Interpreting information from a complex graph that requires some decisions on what features of the graph need to be considered and how information from the graph can be aggregated is a Level 3. Caution is warranted in interpreting Level 2 as only skills because some reviewers will interpret skills very narrowly, as primarily numerical skills, and such interpretation excludes from this level other skills such as visualization skills and probability skills, which may be more complex simply because they are less common. Other Level 2 activities include explaining the purpose and use of experimental procedures; carrying out experimental procedures; making observations and collecting data; classifying, organizing, and comparing data; and organizing and displaying data in tables, graphs, and charts.

Mathematics Depth of Knowledge Levels continued

Level 3 (Strategic Thinking) requires reasoning, planning, using evidence, and a higher level of thinking than the previous two levels. In most instances, requiring students to explain their thinking is a Level 3. Activities that require students to make conjectures are also at this level. The cognitive demands at Level 3 are complex and abstract. The complexity does not result from the fact that there are multiple answers, a possibility for both Levels 1 and 2, but because the task requires more demanding reasoning. An activity, however, that has more than one possible answer and requires students to justify the response they give would most likely be a Level 3. Other Level 3 activities include drawing conclusions from observations; citing evidence and developing a logical argument for concepts; explaining phenomena in terms of concepts; and using concepts to solve problems.

Level 4 (Extended Thinking) requires complex reasoning, planning, developing, and thinking most likely over an extended period of time. The extended time period is not a distinguishing factor if the required work is only repetitive and does not require applying significant conceptual understanding and higher-order thinking. For example, if a student has to take the water temperature from a river each day for a month and then construct a graph, this would be classified as a Level 2. However, if the student is to conduct a river study that requires taking into consideration a number of variables, this would be a Level 4. At Level 4, the cognitive demands of the task should be high and the work should be very complex. Students should be required to make several connections—relate ideas *within* the content area or *among* content areas—and have to select one approach among many alternatives on how the situation should be solved, in order to be at this highest level. Level 4 activities include designing and conducting experiments; making connections between a finding and related concepts and phenomena; combining and synthesizing ideas into new concepts; and critiquing experimental designs.

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