Problem Solving

A Research Based Unit of Study for High School Teachers

Rhode Island Department of Education
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Section 1: Purpose and Design

Problem solving plays an important role in mathematics education, and, like other process standards, should be integrated throughout the various content strands (i.e., Number and Operations, Geometry and Measurement, Functions and Algebra, Data, Statistics, and Probability) and is intended to span across grades. *Problem Solving* is one unit of study in a series of units designed around the big ideas in high school mathematics.

The rationale and basis for problem solving will be briefly discussed in Section 4, but is well established and should be explored through the references in Section 10. The focus of this unit of study is on an introduction to the problem-solving process. The purpose of this unit of study is to support quality instruction by increasing teachers' mathematical knowledge primarily through examining research findings related to students' understandings of the problem-solving process, and examining teaching recommendations based upon those findings. When adapting the ideas in these units of study to the classroom, teachers should purposefully design their lesson plans to incorporate three important principles of learning as identified by the National Research Council in *How Students Learn – Mathematics in the Classroom* and described in Table 1.1 on the following page (Donovan & Bransford, 2005). It should also be noted that this unit is not meant to supplant current curricula materials but rather to be used in conjunction with them. The three principles described in Table 1.1 are modeled throughout this unit of study. The Essential Questions in Section 3 seek to engage preconceptions, the various sections throughout the unit connect procedural knowledge to conceptual knowledge, and there are checkpoints along the way to monitor and reflect on your progress. Additionally, Section 8 contains exercises that will allow you and your colleagues to reflect upon your instructional programs and curricular materials. Since problem solving is such a vast topic, it is recommended that you explore the resources in the reference section and spend time discussing strategies and various solutions to the Essential Questions. It may be beneficial to have a skilled facilitator for the unit of study, including the exercises. This facilitator should have a deep mathematics content background, along with a teaching mathematics background.

Each unit of study begins by examining the Grade-Span Expectations that are pertinent to the particular unit of study. Following the identification of the expectations related to the unit of study, you will answer some essential questions. It is recommended that you answer these essential questions individually prior to reading subsequent sections. The essential questions will help frame the mathematical ideas of the unit. Exercises appear throughout the remaining sections. These exercises are imbedded within the section rather than at the end of the section and are intended to be solved and discussed as you are working through the section. Section 7 is devoted to examining NECAP released items and the student work that is available for these items. This section is subsequent to the sections that discuss research findings. So, once you reach this section, you will be able to make connections between the research and the NECAP items and identify typical student misconceptions.

Additionally, it is worth noting that students need opportunities to work collaboratively, and to share and present ideas. Their understandings should be challenged and students should be allowed to build and construct knowledge for themselves. Students should actively engage in mathematics. Teachers should carefully guide this work, and should be cautious about just telling students the 'answers.' Teachers should ask students to explain how they know and allow them to share multiple ways to solve problems. Teachers need to continually probe students' understandings, especially by allowing students to explore new ideas on their own. Teachers need to resist modeling a few dozen low-level problems (i.e., not cognitively challenging) for students and then asking them to solve far too many homework problems all of which can be matched to one of the modeled problems.

Table 1.1 – Principles of Learning

Principle	Description
Principle 1 – Engaging Preconceptions	Students come to the mathematics classroom with ideas about the structures of mathematics and informal understandings. If their preconceptions are not engaged and if there is no bridge between informal and formal understanding, students may have difficulty learning new ideas and may continue to revert to their preconceived notions.
Principle 2 – Connecting Procedural/factual knowledge and Conceptual Understanding	Procedural knowledge and conceptual understanding must be balanced. When one places too much emphasis on procedural fluency the result is a lack of understanding in how the procedures work. Whereas, when one places too much emphasis on conceptual knowledge, often students lack the ability to perform the procedures in an efficient way. Teachers must help students build and connect ideas and organize knowledge into networks. It is important to discuss various solution methods and why they work and make connections among them.
Principle 3 – Self Monitoring	Students need to be afforded the opportunity to think about their own learning and assess their own mathematical progress. Eventually, such assessment opportunities will be internalized and students will begin to monitor their own progress.

Exercise 1.1 Take some time to complete the following graphic organizer in your journal. You should use this organizer throughout the unit of study and continually add to it. If you are working with colleagues, it is recommended that you each complete your own graphic organizer and post it on chart paper so that you can learn throughout the unit from the work of your colleagues. It is okay if there are portions of the graphic organizer that you can not initially fill out. Again, you should continually return to this graphic organizer as you complete the various sections.

Problem Solving is…	**Things I want to learn about problem solving or integrating problem solving into the curriculum include…**
Students' conceptions and misconceptions about the problem solving process include…	**Teaching strategies for dealing with students' difficulties with problem solving include…**
Various problem-solving techniques include…	**Things I learned include…**

Section 2: Connecting to the Grade-Span Expectations

One of the goals of this unit of study is to help you become more familiar with the Grade-Span Expectations dealing with problem solving. In order to achieve this goal, you will spend some time looking through the Grade-Span Expectations to find the standards related to problem solving.

In Section 7, you will spend time analyzing released NECAP items and student work associated with these items. Please complete Exercise 2.1 before reading on.

Exercise 2.1 Locate the GSEs that deal with problem solving. Make certain that you are considering both state and local GSEs.

Please refer back to the *About the Mathematics GSE* page at the beginning of the GSEs if necessary to recall how bolding and underlining are used in the document. On the following page, assessment GSEs appear in bold boxes; all others are local GSEs.

While problem solving is integrated throughout the content GSEs (i.e., those GSEs contained in the content strands), this unit of study will focus on the following GSEs identified in the process strands (New Hampshire Department of Education & Rhode Island Department of Education, 2006).

High School

M:PRP:HS:1 **Students will use problem-solving strategies to investigate and understand increasingly complex mathematical content** and be able to:

- Expand the repertoire of problem-solving strategies and use those strategies in more sophisticated ways.
- Use technology whenever appropriate to solve real-world problems (e.g., personal finance, wages, banking and credit, home improvement problems, measurement, taxes, business situations, purchasing, and transportation).
- Formulate and redefine problem situations as needed to arrive at appropriate conclusions.

M:PRP:HS:2 **Students will use mathematical reasoning and proof** and be able to:

- Expand the repertoire of proof techniques and use those techniques in more sophisticated ways.
- Use informal and formal reasoning and proof to explain and justify conclusions.
- Formalize mathematical arguments through the use of deductive reasoning.
- Use the principle of mathematical induction.
- Use reasoning and proof throughout classroom discussions independent of the mathematical topic being studied.
- Recognize how reasoning and proof influence the structure of mathematics.

M:CCR:HS:1 **Students will communicate their understanding of mathematics** and be able to:

- Explain and justify their thinking and develop increasingly sophisticated questions for given problem-situations.
- Critique and follow the logic of arguments presented within mathematics and across disciplines.

M:CCR:HS:2 **Students will create and use representations to communicate mathematical ideas and to solve problems** and be able to:

- Choose appropriate representations and mathematical language (e.g., spreadsheets, geometric models, algebraic symbols, tables, graphs, matrices) to present ideas clearly and logically for a given situation.
- See a common structure in mathematical phenomena that come from very different contexts (e.g., the sum of the first *n* odd natural numbers, the areas of square gardens, and the distance traveled by a vehicle that starts at rest and accelerates at a constant rate can be represented by functions of the form $f(x) = ax^2$).
- Find representations that model essential features of a mathematical situation (e.g., cost of postage can be modeled by a step-function).
- Use representations as a primary means for expressing and understanding more abstract mathematical concepts.

M:CCR:HS:3 **Students will recognize, explore, and develop mathematical connections** and be able to:

- Explain in oral or written form how mathematics connects to other disciplines, to daily life, careers, and society (e.g., geometry in art and literature, data analysis in social studies, and exponential growth in finance).
- Explain multiple approaches that lead to equivalent results when solving problems.

Section 3: Essential Questions

Essential questions help you to begin to think about the mathematics that will be the focus of this unit of study. You are encouraged to think deeply about these questions and to work them independently before discussing your thoughts with colleagues and before reading subsequent sections. It is also recommended that you keep a journal that contains your work on these problems and the problems throughout this unit of study. You are encouraged to use PEN in your journal so that you can not easily erase your work. Even though this is contrary to what many mathematics teachers ask students to do, using pen will allow you to go back and reflect on your initial thoughts, analyze any errors that you have made, and see how your learning has developed. (This is a suggestion that Tim Kurtz, NH State Assessment Director, has passed on to me that I try to share whenever possible. As a teacher, if you require your students to use pen, you will be able to easily identify what students were thinking when working problems and any errors made by students. This will facilitate your efforts in addressing students' preconceptions and misconceptions and will allow students to monitor their progress – See Table 1.1.) Some of these questions are intentionally vague in some areas; the reasons will be apparent when one works through the remainder of the sections. Many of these questions will be discussed throughout various sections of this unit of study and will serve to illustrate particular aspects of the problem-solving process. Please refrain from looking at the answers to the essential questions (Section 9) until working through the entire unit of study. Some of these problems may take some time to develop solutions for, and you may want to revisit them often – you can safely read ahead and continue to work on the essential questions throughout the unit. It is advised that you discuss your work and methods with colleagues and share multiple approaches. Additional questions will be posed throughout the various sections. Finally, you should note that each of these problems is carefully chosen to illustrate particular aspects of the problem-solving process. In order for students to embrace problem solving, they must find the problems interesting and relevant. *These problems are not necessarily designed for your students, but rather for you – to allow you to engage in the same problem-solving process that you wish for your students to engage in*. Section 6 will discuss teaching problem-solving to students. If you have seen or solved some of the essential questions recently you probably won't be engaging in the genuine problem-solving process, but you should reflect on how you first solved the problem, any difficulties that you had along the way, any particular strategies that you used, and make an attempt to write out all the details of how you solved the problem. Section 8 will contain links to resources for additional problems that may be of interest to your students. Again, the focus here is to get you to engage actively and genuinely in the problem-solving process.

> "*The first one in the classroom to become a problem solver must be the teacher*." (Wilson, Fernandez, & Hathaway, 1993).

[Note: While the Essential Questions are organized by a particular content strand, it is certainly the case that many of them span across strands – these are just broad categories. Additionally, many of these problems are fairly well known problems (i.e., many of these problems are seen as classics in problem solving and appear in many sources). For most problems, a source is listed to allow you to explore the problem further. We also document whether the version of a particular problem presented here is adapted from the version in the source.]

Some Number and Operations Problems

Essential Question 1 Let *p* be a positive prime number greater than 3. Show that p^2 leaves a remainder of 1 when it is divided by 12.

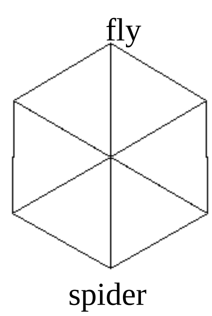
Essential Question 2 Suppose you created a list of all fractions, excluding equivalent fractions (e.g., ½ would only appear in the list once). Without converting the fractions to their decimal expansions, how could you determine which fractions have decimal expansions that terminate and which fractions have decimal expansions that repeat? Explain.

Essential Question 3 Show that for every natural number *n*, there exists a succession of *n* natural numbers which doesn't contain a prime.

Essential Question 4 How many positive divisors are there of 2160?

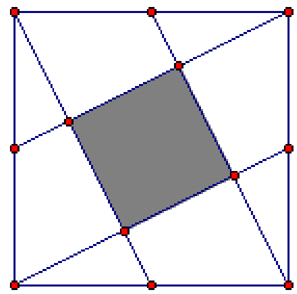
Some Geometry and Measurement Problems

Essential Question 5 A spider and a fly are each at a vertex of a 2-inch cube as shown below.



If the spider walks along the surface of the cube, what is the shortest distance the spider can walk to reach the fly?

Essential Question 6 Given a square with sides of length *s* and line segments connecting the vertices of the square with the midpoints of the sides of the square as shown below, find the ratio of the area of the shaded square to the area of the original square.



(Adapted from Wilson, n.d.)

Essential Question 7 Find all right triangles with sides of integer lengths whose areas and perimeters are numerically equal.

Essential Question 8 Given any triangle, draw a square in the triangle such that each vertex of the square coincides with a side of the triangle. Two of the vertices should lie on one side of the triangle and each of the other sides of the triangle should contain one vertex of the square.

(Adapted from Polya, 1945)

<u>Some Function and Algebra Problems</u>

Essential Question 9 If *a*, *b*, *c*, *d*, and *e* are positive numbers, show that

$$\frac{(a^2+1)(b^2+1)(c^2+1)(d^2+1)(e^2+1)}{abcde} \geq 32 .$$

(Adapted from Schoenfeld, 1983)

Essential Question 10 Determine the sum of each of the following finite series.

a) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+...+\frac{1}{n\cdot(n+1)}$

b) $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+...+\frac{n}{(n+1)!}$

(Adapted from Schoenfeld, 1983)

Essential Question 11 Let $p(x)$ and $q(x)$ be polynomial functions with “reversed” coefficients. That is,

$$\begin{aligned} p(x) &= a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x^1 + a_0 \\ q(x) &= a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-2} x^2 + a_{n-1} x^1 + a_n \end{aligned}, \text{ where } a_n \neq 0 \neq a_0$$

What is the relationship between the zeros of $p(x)$ and $q(x)$? Explain.

(Adapted from Schoenfeld, 1983)

<u>Some Data, Statistics, and Probability Problems</u>

Essential Question 12 Two red chips and one black chip are placed in an envelope. Two chips will be randomly drawn from the envelope without looking. If the colors of the chips are different, the teacher drawing the chips will score one point. If the colors are the same, the students will score one point. Notice that the game is not a fair game. In order to make the game fair, you are allowed to add a single chip (a red chip or a black chip) to the envelope to make the game fair. What single chip should you add? Explain.

(Adapted from Posamentier & Hauptman, 2006)

Essential Question 13 The probability of breast cancer is 1% for women of age forty who participated in a routine screening. Additionally, 80% of the women who actually have cancer will receive a positive test and about 10% of women who don’t have cancer will receive a positive test. Given that a woman in this age group tested positive for breast cancer, what is the probability that she actually has breast cancer? Explain.

(Adapted from Krynski & Tenenbaum, n.d.)

Essential Question 14 Suppose you were to spin a penny 40 times. How many times would you expect the penny to land on heads? Explain. How would you go about determining whether or not your prediction was accurate?

Essential Question 15 When finding the least-squares regression line, we seek to minimize the sum of the squared residuals. Why is this method preferable to minimizing the sum of the absolute values of the residuals?

<u>Some Recreational Mathematics Problems</u>

Essential Question 16 A voter with a rowboat faces a difficult dilemma. On one side of a river is Bob Dole, Bill Clinton, and a big bag of burgers. The voter must get Dole, Clinton, and the bag of burgers across the river. Of course, Dole, Clinton, and the bag of burgers can not row themselves across the river. Furthermore,

- The voter’s boat is only large enough to hold herself and one person or object;

- Dole can’t be left alone with Clinton otherwise Dole (the former war veteran) will beat up Clinton; and
- Clinton can’t be left alone with the burgers otherwise Clinton will eat the entire bag (leaving no burgers for anyone else).

Is it possible to get Dole, Clinton, and the burgers across the river? If so, carefully explain how; if not, carefully explain why.

(Adapted from Burger & Starbird, 2005)

Essential Question 17 Follow the link below to the National Library of Virtual Manipulatives and solve the variations of the counterfeit coin problems (i.e., the 8 coins, 9 coins, 12 coins, and challenge problem): http://nlvm.usu.edu/en/nav/frames_asid_139_g_4_t_2.html?from=query.htm?qt=counterfeit+coins&lang=en.

The next section briefly covers some of the key ideas around problem solving that underlie this unit of study. The section will also discuss some various heuristics that students use when approaching problems and some of the habits of mind that we seek to develop in students.

Section 4: Problem Solving Techniques

The National Council of Teachers of Mathematics (NCTM) describes problem solving as engaging in a task for which the method of obtaining the solution is not known in advance (NCTM, 2000). In a 1977 position paper, the National Council of Supervisors of Mathematics (NCSM) described problem solving as the principal reason to study mathematics. Furthermore, theories that describe how people learn and construct knowledge serve as a foundation for teaching mathematics through a problem-solving approach (D'Ambrosio, 2003). The National Research Council describes problem solving as being central to school mathematics and the site by which the strands of mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition) converge (NRC, 2001). Opportunities to solve problems and use various strategies should be embedded throughout the curriculum and across content areas in natural ways (NCTM, 2000).

Furthermore, by engaging in the problem-solving process (as described below, but not to be confused with problem-solving heuristics – see Table 4.2), students will develop important Habits of the Mind (see Table 4.3) or modes of thought that go beyond the understanding of the content and will be useful in a wide variety of situations.

An important aspect in problem solving, as described above, is involving students in tasks for which the method of obtaining the solution is not known in advance – this requires the development of a 'good' problem. A characteristic of a good problem, as described by NCTM, is the integration of multiple topics and the involvement of significant mathematics. Additionally, good problems will allow students to solidify their understandings of mathematical topics and extend those understanding to build new knowledge.

Often, the problem solving process is presented in a series of steps. But, as you will soon see, this is not within the spirit of true problem solving. Most likely, these steps can be traced to those outlined by Polya in a now famous book called *How to Solve It* that was first published in 1945. Polya outlines problem solving as containing four phases that can be summarized as follows:

- Understand the problem
- Develop a plan
- Carry out the plan
- Look back

Table 4.1 describes these phases in further detail.

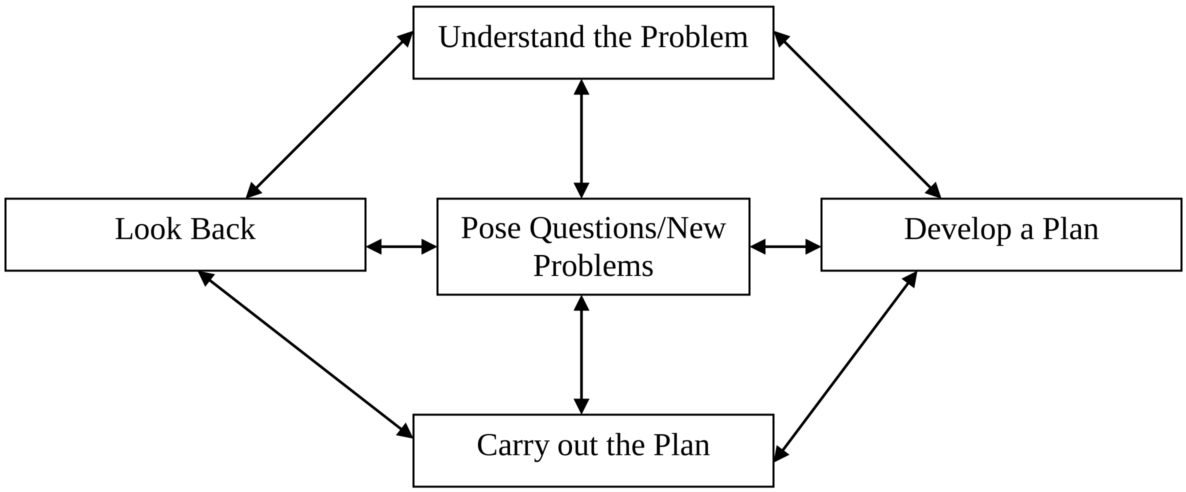
Table 4.1 – The Problem-Solving Process
(Adapted from Polya, 1945)

Problem-Solving Step	Additional Details
Understand the Problem	This step includes understanding the unknown(s), the data given in the problem, what the condition is, how the condition links the unknown values to the known values, whether or not the condition is sufficient to determine the unknown. Additionally, a diagram may be drawn with appropriate notation introduced.
Develop a Plan	Once the difficult work of clearly understanding the problem is complete, students often think that developing a plan should be an easy task. Often this can be quite difficult and takes some perseverance. It is in this stage that you should take care to give the student sufficient help so that he or she believes that success on the problem is possible, but you need to be careful to leave enough work for the student such that the student builds and constructs knowledge for himself or herself. When developing a plan, students may pull from any of the heuristics that they have studied (see Table 4.2); however, the particular problem may be more accessible with a strategy that is an adaptation of those studied. While guiding students to develop a plan you might consider asking any of the following questions or adaptations of them. • Have you seen the same problem before, but in a slightly different way, or a similar problem? • Do you know of a related problem? • Do you know a useful theorem? • Do you know of a familiar problem that has the same unknown? If so, can its solution be useful?

	• Look at this related problem that you previously solved. Can you use a similar method? • Should you introduce some auxiliary elements? • Can you restate the problem, perhaps in a slightly different way? • Can you think of an easier related problem? • Can you solve part of the problem? • What happens if you keep only part of the condition? • Can you determine another useful result from the given data? • Did you use all the data? • Did you use the entire condition?
Carry out the Plan	Once you begin to carry out the plan, you should check each step along the way making certain that each intermediate result is reasonable. Often, carrying out the plan leads to the need to prove many intermediate results.
Look Back	Looking back involves checking the result and the argument, considering whether or not the result can be obtained in a different way, and determining whether or not the result may be useful in solving some other problem.

Unfortunately, these stages are often presented in textbooks in a linear fashion, rather than in a flexible way as described by Polya, and, hence, are not consistent with the spirit of problem solving that Polya was advocating – building flexible thinkers. Additionally, these steps, when outlined in a linear fashion, tend to emphasize the importance of obtaining an answer (Wilson, Fernandez, & Hadaway, 1993). Consequently, the importance of the process is unintentionally deemphasized. Furthermore, when describing the looking back phase (which occurs subsequent to obtaining the answer), Polya states that a good teacher should realize that no problem is completely exhausted – there always remains something to do (Polya, 1985). This 'something' may be a better understanding of the problem, improving the solution, looking for connections, deriving the result in a different way, determining whether or not the result can be generalized, or the like. Polya's descriptions of the four phases above lead to a dynamic cyclic interpretation, rather than a linear interpretation, of the phases which are demonstrated in Figure 4.1 and have been used successfully in problem-solving courses at the University of Georgia (adapted from Wilson, Fernandez, and Hadaway, 1993).

Figure 4.1 – The Problem-Solving Process



Wilson, Fernandez, and Hadaway point out that the model above is not a theoretical model, but rather a framework for discussing pedagogical, curricular, instructional and learning issues that are involved with the spirit and goals of genuine problem solving (Wilson et al., 1993). The key is to understand that one needs to be flexible when working with the phases outlined above. Problem solving should not be viewed as a series of steps. This idea is important to keep in mind as you are working throughout this unit of study. While we will often examine various strategies associated with problem solving, the goal is to develop independent, flexible thinkers that pull from the various strategies as needed. In fact, we should challenge students' understandings of the various strategies. Often, students will think that a particular strategy that they learn should easily apply in situations that seem similar to the problems where those strategies are modeled. While this may be the case, it certainly isn't always the case, and students should have the opportunity to experience such challenges. This idea will be presented in exercises in this section.

In general, one is trying to move in a clockwise fashion around the model illustrated in Figure 4.1 (that is, one is not moving from the 'developing a plan' stage directly to the 'looking back' stage without first carrying out the plan). However, as the model illustrates, student may understand the problem and move into the planning phases, but consequently gain a deeper understanding of the problem. This may occur by posing questions or restating the problem. Hence, students may revisit the understanding the problem phase. In fact, we might argue that understanding the problem is the most important phase – if you do not have a clear understanding of the problem, how can you develop and carry out a plan for solving the problem? In fact, it is often the case that we see students trying various computational methods without a clear path as to where they are going, and we can each think back to times where we have tried to prove mathematical results without fully understanding the statement of the theorem. Hence, sufficient time should be spent on this phase. Posing questions or new problems can occur during any phase and is central to the process. These new problems may lend an understanding to the previous problem or a deeper understanding of previously studied concepts. Many of these problems will be interesting to explore, but the teacher will have to balance which ones to explore based upon the goals inherent in the original problem. Since many of these new problems will be ones that the teacher hasn't originally planned for, they can provide the teacher the perfect opportunity to model the problem solving process for his or her students. Too often the problem solving process isn't effectively modeled by teachers. As educators, we prepare for our classes and often don't make many mistakes while in front of the class (typically, only superficial ones). Students tend to believe that mathematics is never a struggle for their teachers. In reality, allowing students to see us struggle, but successfully model the process, can be quite beneficial to students.

Example 4.1 – Modeling the Cyclic Nature of the Problem-Solving Process

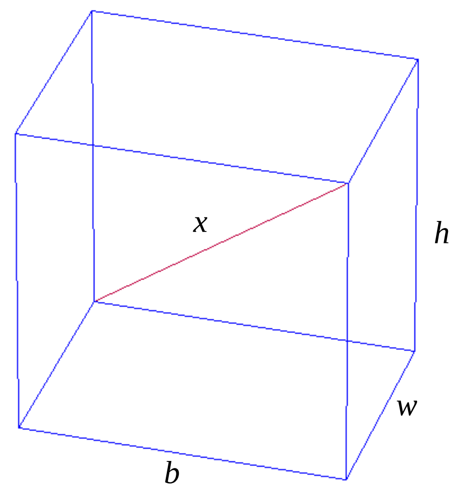
This example will summarize a problem discussed by Polya in *How to Solve It* (It is recommended that you see the source for a more thorough discussion). The example is purposely simple in nature, but illustrates the cyclic nature of the problem-solving process and the spirit of Polya's phases. Before reading on, you should attempt to think about how you can guide students through the process described in Figure 4.1 for this problem. For example, what questions might you ask students to carefully guide their understanding of the problem? What additional questions might this problem lead to? Take a moment to reflect on your thoughts in your journal after reading the problem. [In order for students to engage successfully in this task they would have to be familiar with the Pythagorean Theorem.]

> Determine how to find the length of a diagonal of a rectangular parallelepiped given the lengths of the base, width, and height.

Guiding students to understand the problem goes beyond a surface understanding of the problem statement. It also includes understanding what the unknown is, what the given values are, and the condition that links the unknown value to the known values. Furthermore, we must understand whether or not that condition is sufficient to determine the unknown. We illustrate each of these in the following discussion.

Understand the Problem

In this case, you should clearly begin by understanding what a rectangular parallelepiped is. You might point out that the classroom is a rectangular parallelepiped (of course, not all classrooms are) and the students might realize that they are trying to determine how they might find the length of a diagonal of the classroom by knowing the length of the base, length of the width, and length of the height. Here you have helped the students make a concrete connection. Then, you can ask the students to suggest appropriate notation for the known values and unknown value. For discussion purposes, we will assume that students might suggest denoting the lengths of the base, width and height b, w, and h, respectively. And, we will call the unknown (i.e., the length of a diagonal) x. Then, students should be guided to see the condition that links the known values to the unknown (i.e., in this case, simply that x is the diagonal of a parallelepiped whose base measures b, width measures w, and height measures h).



Next, students should be engaged in a discussion as to whether or not the condition is sufficient to determine the unknown. While at this state, students have not developed a plan to determine the length of the diagonal, and it may initially seem that students can not determine if the condition is sufficient, we note that once the lengths of the base, height, and width of a rectangular parallelepiped are known, the length of the diagonal is determined – that is, if you fix the base, width, and height there can only be one possible length for the diagonal (i.e., the condition is sufficient to determine the unknown). So, from an intuitive perspective, students can determine that the condition is sufficient to determine the unknown.

Develop a Plan

Developing a plan, in many cases, can take a great deal of time. Obtaining the solution depends upon the development of the plan. A plan is developed once you have a clear path of which calculations, computations, constructions, and previously derived results are needed to obtain the solution. As described in Table 4.1, the teacher needs to be careful at this stage to not give away too much of the solution. That is, the teacher should carefully guide the student to a good idea by using questioning techniques like those described in Table 4.1. Asking students if they know of a related problem is a good start. As Polya describes, often there are too many related problems that students have seen to know which are relevant. However, a focus point is to look at the unknown and make certain that the related problems involve the same unknown (i.e., in this case the length of a diagonal). Often, students will not make a connection to a previously related problem. This is when the teacher will guide the student further and ask a more direct question such as, “Have you ever solved a problem where the unknown was the length of a diagonal?” From here, students may develop the idea that the use of the Pythagorean Theorem can help lead to a solution. The teacher may then ask how the students could use the Pythagorean Theorem in a way that involves the unknown (after students have developed the idea to use the Pythagorean Theorem on their own). If students do not see a solution, the teacher then might ask if the students can find a right triangle in the figure that involves the unknown. Here, the teacher may rely on models for help and need to guide the students to drawing in an auxiliary line segment (i.e., in this case, the diagonal of the base from the upper left corner to the lower right corner). However, the teacher should be prepared that this still isn’t sufficient to guide students to the idea and must be prepared with other questions to help students come to this plan (again, the key is not to do the work for the student, but to carefully guide the student to the plan). Once students have reached this stage the teacher should be careful to make certain that students have indeed developed a plan to reach the solution. That is, the student may not have realized that the diagonal of the base needs to be determined by using the Pythagorean Theorem as well. Once students have reached this point, the teacher can be assured that they have developed a complete plan.

Carrying out the Plan

Understanding the problem and developing the plan tend to be much more difficult than carrying out the plan. In this particular step, the teacher should be careful to remind the student to check their work along the way. Additionally, the student may need to be guided to introducing notation for the unknown diagonal of the base, which in this case we will call *y*. From here, the student can apply the Pythagorean Theorem to obtain:

$$y^2 = b^2 + w^2$$

$$x^2 = y^2 + h^2$$

Students at this point may need to be asked questions such as, “What is it that you are trying to find? Did you introduce any auxiliary unknowns? What was your goal? What were your original known values?” That is, questions that will help the student recall that they are trying to express the unknown value, *x*, in terms of the known values *b*, *w*, and *h*. From here, students can then make a substitution to obtain:

$$x^2 = b^2 + w^2 + h^2, \text{ or}$$
$$x = \sqrt{b^2 + w^2 + h^2}.$$

You may be thinking that you don’t have the class time to engage in such a process, but the more you do so, the less time it will take as students will become more effective problem solvers. In the long run, students will be developing independent thoughts and solving more challenging problems with greater ease.

Looking Back

Polya states that good teachers should realize that no problem is completely exhausted and should work to impress this belief upon his or her students (Polya, 1945). That is, when looking back, one can always strive to gain a deeper understanding of a solution, further insights into the problem, and look for further ways to verify that the solution is correct. Once students obtained a way of determining x from the known values, they will often look to move onto the next problem. However, reflecting on the problem can serve to consolidate the student's knowledge and lead to other insights. Additionally, mistakes could be made along the way, so the teacher should ask the student if he or she can check the results. We might also ask the student if he or she could derive the result differently or see an easier way to solve the problem. Students should look back to see if they used all the data.

Exercise 4.1 Before reading on, describe some ways in which you might verify that the solution to the parallelepiped problem is correct that will help you gain deeper insights into the problem.

To verify whether or not the solution to the above problem is correct, and to gain a deeper understanding of the problem, we might look at any of the following questions. Does the result remain unchanged if we interchange b, w, and h? Is the result of this problem analogous to the two-dimensional problem? What happens as the height decreases? Does this result make sense (i.e., trying to see that as the height goes to zero the formula becomes the familiar version of the Pythagorean Theorem from the plane)? As the height increases, what should happen to the length of your diagonal? Does your result show this? Students may then be asked if they can use the result for some other problem and could be encouraged to create their own applications for which the result could be used.

Polya cautions that during the above process the teacher must be careful to ask good questions. For example, asking the question, "Do you know of a related problem?" is much different from asking students, "Can you apply the Pythagorean Theorem?" The former carefully guides students to the development of the idea on their own, whereas, the latter gives away the idea that the student is trying to discover in the developing the plan phase.

Exercise 4.2

Create a problem that is relevant to your class that will allow students to engage in the problem-solving process. Anticipate some of the difficulties that students will have solving the problem and create a series of 'good' questions for each of the four phases that will help guide students to the solution and the appropriate reflection. Share your problem and questions with colleagues.

Example 4.2 – Illustration of Looking Back

Take a moment to revisit Essential Question 2 from Section 3.

> **Essential Question 2** Suppose you created a list of all fractions, excluding equivalent fractions (e.g., ½ would only appear in the list once). Without converting the fractions to their decimal expansions, how could you determine which fractions have decimal expansions that terminate and which fractions have decimal expansions that repeat? Explain.

While solving this problem, it may be easiest to start with what we are familiar with. That is, if we have a decimal expansion that terminates, it is fairly straightforward to write that decimal expansion as a fraction. That is, suppose that the decimal expansion contains n digits, than we can write the fraction in the form $\frac{p}{q}$, where $q = 10^n$ (some details are being skipped here and are left to the reader). From here, this fraction may be reduced and we start to see that denominators of fractions that terminate look like some factor of some power of 10. Additionally, any power of 10 only contains 2's and 5's in its prime factorization; hence, the reduced denominator must only have 2's and 5's in it prime factorization. While these observations now lead us to a solution of the original problem, there are still many unexplored and interesting questions that we might ask. For example, the task asked us to suppose that we could list all fractions, so we first might ask whether or not this is possible and if so, how would we do it? From here, this question will open up a whole new area of questions connected to different sizes of infinity and cardinality. Take some time to explore this idea.

As mentioned earlier, the problem-solving process is often described as a set of strategies that might be used when developing and carrying out the plan. In this unit, we make the distinction between the process (as described in Figure 4.1) and the strategies, and refer to the strategies as heuristics. That is, problem solving heuristics can be described as strategies that enable students to work through the problem-solving process. The strategies are independent of the content topic. It is important to note that these are just specific strategies. And, again, students should experience mathematical problems that challenge these strategies. That is, once a student learns a particular strategy, they might assume that that strategy is appropriate for all similar problems – this understanding should be challenged so students develop flexibility when solving problems. See Example 4.3. The heuristics listed in Table 4.2 might be organized into the various components of the problem-solving process (e.g., drawing a diagram may aid in understanding the problem). It is important to note that this is just a short list of heuristics and teaching each of these will not guarantee success in problem solving. Each of these, along with the problem solving process, need to be constantly modeled for students, so they can learn which of these many heuristics may aid in solving a particular type of problem. Additionally, as with the problem-solving process, many of these heuristics interact with each other and are not intended to be taught in a checklist style.

Table 4.2 – Various Problem-Solving Heuristics
(for a more complete list and detailed descriptions see Polya, 1945)

Heuristic	Brief Explanation
Draw a Diagram	Drawing a diagram should be considered an important strategy for problems that are geometric in nature and also those that are not. Often, the diagram can help us determine how the known values are related to the unknown values.
Examine Special Cases	Examining special cases (e.g., what happens when x is positive) can help uncover important patterns and lend new insight into the problem, including eliminating possibilities.
Eliminate Possibilities	Systematically accounting for, together with eliminating possibilities, are connected to logical deduction. The method of seeking contradictions may be used frequently here.
Solve an Easier Related Problem	Often a problem is too complex to know where to begin. Solving an easier related problem, potentially by examining special cases, can help you understand the more complex problem.
Look for a Pattern	Mathematics is described by some as the study of patterns and the ability to interpolate and extrapolate those patterns. Patterns often underlie the important mathematics in the problem and help to uncover relationships between the variables involved.
Work Backwards	Working backwards requires taking the result and determining what previous position will get you to that result. It is useful in many situations, and is used widely in algebraic thought (e.g., solving equations).

Example 4.3 -- Challenging Students Use of Heuristics
(Adapted from Schoenfeld, 1983)

Take a moment to reconsider part a) of Essential Question 10 from Section 3.

Essential Question 10 Determine the sum of each of the following finite series.

a) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+...+\frac{1}{n\cdot(n+1)}$

When you solved this problem, you may have started by trying some specific cases (since this is a common strategy for these types of problems). For example, you might replace *n* with 1, 2, 3, 4, and 5 to see if some type of pattern will emerge.

n	$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+...+\frac{1}{n\cdot(n+1)}$	**Sum**
1	$\frac{1}{1\cdot 2}$	$\frac{1}{2}$
2	$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}$	$\frac{2}{3}$
3	$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}$	$\frac{3}{4}$
4	$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}$	$\frac{4}{5}$
5	$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}$	$\frac{5}{6}$

Hence, from the pattern, we can see that $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+...+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$, which can be verified by induction. Additionally, we might note that $\frac{1}{k\cdot(k+1)}=\frac{1}{k}-\frac{1}{k+1}$. Thus, the sum collapses to $1-\frac{1}{n+1}=\frac{n}{n+1}$. Through this example, students might learn that they should try some specific cases whenever a variable is involved. This is a good strategy, but may not always be the best approach, as illustrated in the following example.

> In a single elimination chess tournament, opponents are randomly paired to play games. The loser of each game is eliminated from the tournament while the winner goes on. If there are an odd number of players in a particular round, one person automatically advances to the next round without playing a game. If *n* people enter the chess tournament, how many games must be played before the winner is determined?

Take a moment to solve this problem before reading on. You many be tempted to begin this problem in the same manner as the previous problem. That is, starting with a different number of people and determining how many games will be played, as illustrated below.

n	Number of Games Played	Rationale
2	1	A single game will determine the winner.
3	2	Two players will play one game in the initial round with one player automatically advancing. Then a final game will determine the winner.
4	3	Two games will be played in the initial round with two winners moving on. A final game will determine the winner.
5	4	Two games will be played in the first round with one player automatically advancing. The second round will contain the two winners from the first round plus the player that advanced. Again, one player will automatically move forward, with one game being played in the second round. The winner from this game will play the player who advanced in a final game for a total of 4 games played.

Obviously, a pattern begins to emerge and we can see that if *n* players enter the tournament, then *n* – 1 games are played. However, as we complete this process the pattern starts to become obvious – if *n* players enter the tournament there will be *n* – 1 players who do not win each who participated in a single game. Thus, *n* –1 games must be played to determine the winner. Stopping to understand the problem, rather than moving directly to a particular heuristic leads to a simpler solution.

Exercise 4.3 Reflect on situations in your classroom where students have used previously learned techniques that lead to more complicated solutions rather than gaining a full understanding of the problem. Share two specific examples with colleagues (make certain that they first have an opportunity to solve the problems themselves).

Example 4.4 – Illustration of *Solve an Easier Related Problem*

Take a moment to reconsider Essential Question 9 from Section 3.

Essential Question 9 If a, b, c, d, and e are positive numbers, show that

$$\frac{(a^2+1)(b^2+1)(c^2+1)(d^2+1)(e^2+1)}{abcde} \geq 32 .$$

(Adapted from Schoenfeld, 1983)

It is very tempting to begin this problem by trying some type of algebraic manipulation. However, we should begin by asking ourselves what we will accomplish with algebraic manipulation. If the goal of the algebraic manipulation is to bring everything to one side and show that it is positive, this certainly seems to be a difficult approach. However, some manipulation can aid us in understanding the problem and contribute to identifying a heuristic that may be used in the development of a plan. For example, using our knowledge of multiplication of fractions, we could see that the problem is equivalent to:

$$\frac{a^2+1}{a} \cdot \frac{b^2+1}{b} \cdot \frac{c^2+1}{c} \cdot \frac{d^2+1}{d} \cdot \frac{e^2+1}{e} \geq 32.$$

From here, we might make the observation that $2^5 = 32$ and that each quotient looks alike (this might also be observed by looking at the special case where $a = b = c = d = e = 1$). From here, we can try to solve the easier related problem of whether or not

$$\frac{a^2+1}{a} \geq 2.$$

We might realize that we have solved problems like this before when studying quadratic equations, and we are looking to see if $a^2 - 2a + 1 \geq 0$ (since a is positive). From here, there are a number of ways to reason this inequality is true, including that the function $f(a) = a^2 - 2a + 1 = (a-1)(a-1)$ is a parabola opening upward with a repeated zero at 1, or vertex at (1, 0), and is therefore always greater than or equal to zero. Repeating this argument on the other quotients gives the desired result. In this case, looking at an analogous one-variable problem proves to be an effective strategy.

Example 4.5 – Illustration of Eliminating Possibilities & the Interaction of Multiple Heuristics

Take a moment to revisit Essential Question 1 from Section 3.

Essential Question 1 Let p be a positive prime number greater than 3. Show that p^2 leaves a remainder of 1 when it is divided by 12.

Review your strategy on the problem and see if you can solve the problem by accounting for all possibilities and systematically eliminating possibilities before reading on.

Relying on our knowledge of division (i.e., using some previously derived results that are related to our problem) can help us account for all possibilities. Starting in a broad sense, we realize that we are looking at whole numbers that are greater than or equal to 4. And, any whole number greater than or equal to 4 can only leave a remainder of 0, 1, 2, or 3 when divided by 4. That is, the number must be of the form $4n$, $4n + 1$, $4n + 2$, or $4n + 3$, for n a natural number. Since p is a prime number greater than 3 it can not be even, so we eliminate the possibilities that p is of the form $4n$ or $4n + 2$. Thus, p must be of the form $4n + 1$ or $4n + 3$. Now, working backwards, we realize that the problem is equivalent to showing that 12 divides $p^2 - 1 = (p - 1)(p + 1)$. Starting with $p = 4n + 1$, we have $p^2 - 1 = (p - 1)(p + 1) = 4n(4n + 2)$. Since $4n$, $4n + 1$, and $4n + 2$ are three consecutive integers, one must be divisible by 3 and it isn't $4n + 1$, since we are assuming our prime p is of this form. If 3 divides $4n$, then by our knowledge of prime factorization, we can see that 3 must divide n or that $4n$ is some multiple of 12 making $p^2 - 1$ divisible by 12. On the other hand, if 3 divides $4n + 2$, then $4n + 2$ is some multiple of 3 and the product $4n(4n + 2)$ is a multiple of 12 and hence 12 divides $p^2 - 1$. Accounting for the possibility that p is of the form $4n + 3$, in a similar manner, gives us the solution.

The previous portion of this section has focused on the problem-solving process along with various heuristics that may be engaged in the process. As stated earlier, it is important to challenge students' understandings of the problem-solving process, along with their use of heuristics. Equally important is the development of what is known as Habits of the Mind.

It is important to note, as with the problem-solving process, that the development of these habits of the mind does not occur in a linear fashion and necessarily as a direct result of instructing students in each habit (just as teaching a particular heuristic may not lead to flexibility in using the heuristic). These habits of the mind take time to develop and should be integrated throughout the curriculum. Development of the habits of the mind is often a by-product of their learning mathematics through a problem-solving approach (Levasseur & Cuoco, 2003). Table 4.3 lists a few of the Habits of Mind that we should be striving to develop in mathematics students. (Please note that this list is not intended to be exhaustive. For a more complete list, please see *Mathematical Habits of Mind*, *Understanding University Success*, and www.habitsofmind.org.)

Table 4.3 – Habits of the Mind
(Adapted from Levasseur & Cuoco, 2003 and Conley, 2003)

Habit of the Mind	Description
Guessing/Risk Taking	Making a reasonable guess is an important part of the problem-solving process and should be encouraged. Often a guess can lead one along the correct path or, in the case that the guess is not correct, can help you discover an important aspect of the problem. Students should understand that when they make mistakes it forces them to

	reevaluate their work. Determine what lead to the mistake – were there correct elements in the approach? How could he or she go about the problem differently?
Challenging Solutions	Once students believe that they have found a solution to a problem, even a correct one, they need to develop the habit of challenging those solutions. This will often lead to a deeper understanding of the problem and quite possibly reveal a more complete solution (as illustrated in Example 4.6).
Looking for Patterns	You will notice that some habits of the mind overlap with problem-solving heuristics. See the description for looking for patterns under Table 4.2.
Conserve Memory	This habit of mind has to do with memorizing as little as possible. Building conceptual understanding will help to eliminate excessive memorization (and an in-depth understanding can help with facts that are memorized – in order to make sense of them).
Technology	Value the use of technology, but recognize the appropriate use of technology and that technology does not replace knowledge of basic facts and skills.
Use Alternative Representations	Students should seek to represent problems in various ways, including in words, symbols, tables, and graphs and develop flexibility moving between representations.
Positive Disposition & Perseverance	We want students to value and appreciate mathematics, recognize that we often learn from our failures, and develop a positive disposition towards doing mathematics, and the confidence needed to solve increasing complex mathematical tasks and problems unique to their environments.
Curiosity/Making Conjectures	One of the by-products of a course designed around problem-solving, where students are engaged in doing mathematics and creating mathematical ideas, is the development of curiosity for mathematics and mathematical ideas new to the students. This curiosity is often lost in plug-n-chug oriented classrooms. When

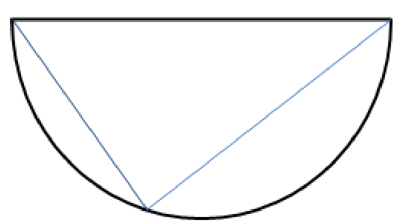
	procedures are taught without conceptual understanding, students are forced to memorize and they mistake the ability to memorize mathematical facts and algorithmic procedures for understanding. Building mathematical curiosity requires allowing students to explore their own conjectures and build and construct ideas on their own.

Example 4.6 Levasseur and Cuoco, in *Teaching Mathematics through Problem Solving*, demonstrate through the following problem how guessing can be an important strategy, along with the important habit of mind of challenging solutions, even correct ones.

Which triangles can be divided into two isosceles triangles? (Levasseur & Cuoco)

Make certain to attempt to solve the problem before reading on. Often guessing can lead one in the correct direction. And, even if guessing doesn't lead to the correct solution it may reveal important information along the way. When guessing, it is important to try to make a reasonable guess – perhaps based on previous experience. It is also important to reflect on what you might learn if you pursue the guess. That is, you shouldn't just wildly go down paths without first reflecting on what you might learn by pursuing that path. For example, why go through a lengthy algebraic process if you are not likely to learn much from it?

A reasonable guess would be to start by looking at an isosceles triangle or other types of special triangles that we might be familiar with. An isosceles triangle leads to a dead end, but may encourage us to look at different types of special triangles. When we consider a right triangle, we might recall that right triangles can be formed by being inscribed inside semi-circles, like in the figure below.



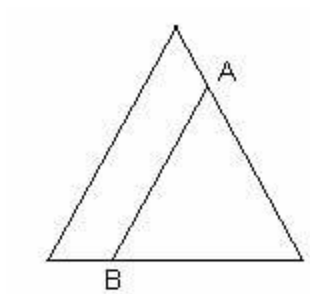
Drawing a radius to the vertex of the right angle convinces us that all right triangles can be divided into two isosceles triangles. At this point, we might be satisfied that we have solved the problem since we have a correct solution; however, as mentioned earlier we should challenge this solution. For example, we might wonder if there are any other solutions. That is, have we found them all? Are there other ways to interpret the problem? Do we notice any other interesting results? To help illustrate exploring the problem further and challenging correct solutions, consider the following vignette taken from an actual conversation between two people working to solve this problem, along with the different strategies they employ, including working to first understand the problem (we will refer to the two people as R and P – also note that this conversation happened over a series of e-mails).

R: Hey P, consider this interesting problem – Which triangles can be divided into two isosceles triangles? I have some ideas but I'm wondering what you are thinking.

P: Hey R, I'm not sure that I understand the problem. Before I start working on it, I want to have a clear understanding of what we mean by "divided." One issue with this problem is there's no mathematical definition attached to the word "divided" in the sense of a triangle.

R: Hey P, I agree that we might interpret "divided" differently. Let's explore different interpretations since the ambiguity actually makes this problem more interesting than what we might have originally thought. What did you have in mind?

P: See below. Does AB "divide" the triangle into two isosceles triangles? We can't really say, because "divided" is not well-defined.



R: That's interesting and makes us rethink our initial thought about starting with an isosceles triangle leading to a dead end. I was thinking about "divided" in the sense that the two interior regions created would need to be isosceles triangles. So, if I worked backwards and think about sticking the two interior regions together to form the original triangle then:

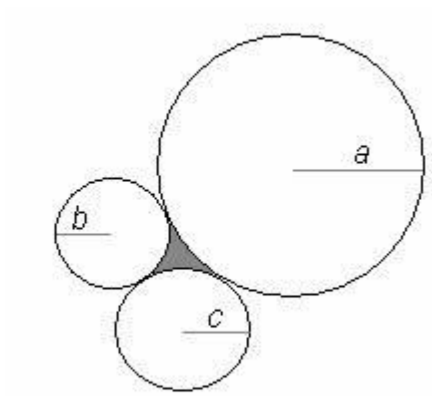
- The line of dissection could be a leg in both triangles (which I can show leads to the right triangle case that we talked about earlier);
- The line of dissection could be a leg in one triangle and a base in the other triangle (which leads to a new interesting set of solutions);
- The line of dissection could be a base in both triangles (well, this couldn't happen since we wouldn't be able to paste these together to form the original triangle).

Have I exhausted all the possibilities or do you have something else?

P: Hey R, I did notice (and prove) that if one angle is three times the measure of another in a triangle, then I can form two isosceles triangles. What do you think about that?

R: Hey P, that's interesting, but I believe I can show that if this happens than the line of dissection is a leg in one triangle and a base in the other. So, while this is interesting and a nice new observation, do we have any other solutions to think about?

P: Hey R, this problem is interesting, but I think we have exhausted it for now, but we can keep thinking about it. How about this one? Find the area of the shaded region below.



While the above conversation occurred over e-mail, it highlights the importance of understanding the problem and employing multiple heuristics when problem solving, including challenging solutions even when they are correct.

Exercise 4.4 Take some time to explore the problem-solving process and problem-solving heuristics more thoroughly by examining some of the references in Section 10 (e.g., *How to Solve It*, *Crossing the River with Dogs*, and *Teaching Mathematics through Problem Solving*). Then, revisit your work in your journal and reflect upon your first attempts to solve the essential questions in Section 3. Take time to discuss various approaches to the problems in Section 3 with your colleagues.

Section 5 will examine some of the research related to problem solving, including research related to students' understandings of the problem-solving process, research related to curricula materials and problem-solving, and research related to classrooms that focus on problem solving.

Section 5: Research Snippets Related to Problem Solving

This section briefly examines some research related to students' understandings around problem solving and also research related to curricula materials and classrooms that focus on problem solving. The next section will examine some instructional strategies that can help teachers foster students' understandings of problem solving and some of the challenges described in this section.

When researchers study problem solving, they typically devise tasks for students to work on and then observe students as they solve and talk about those tasks. Some findings from these observations are summarized below in Table 5.1. Of particular importance is the finding related to students familiarity with the problem-solving process. Despite the National Council of Teachers of Mathematics emphasis on the process standards (e.g., communication, connections, problem-solving, representations), along with a similar emphasis in state standards, and teachers reporting that their curricula is aligned to these standards, researchers find that students are not well-acquainted with the problem-solving process.

Table 5.1 – Research Related to Students' Understandings of the Problem-Solving Process (Wilson, Fernandez, & Hadaway, 1993)

Familiarity with the Problem-Solving Process	Research suggests that students are not acquainted with the problem solving process.
Knowledge Base	A good content knowledge base is needed to solve problems. Novices attend to surface features of problems.
Categorization of Problems	Successful problem solvers are likely to categorize problems based on their underlying mathematical similarities (see earlier discussion – do you know a similar problem?).

In *Teaching Mathematics through Problem Solving*, Stein, Boaler, and Silver summarize research findings related to the following two questions.

- How does the use of curriculum materials that promote teaching mathematics through problem solving affect students' learning of mathematics?
- What happens inside classrooms where effective problem-solving approaches are employed?

What follows in Table 5.2 and Table 5.3 are summaries of these findings.

Table 5.2 – Research Findings related to Curriculum Materials (Stein, Boaler, & Silver, 2003)

Greater Conceptual Understanding	One of the consistent findings across research studies is that students who are taught using reform curricula (e.g., a National Science Foundation curriculum such as the Interactive Mathematics Project) generally show a higher level of conceptual understanding and greater ability in problem solving when compared to students taught using traditional curricula.
Same Level of Performance on Skills & Procedures	One of the frequent criticisms of reform curricula is that there isn't enough emphasis on basics skills and procedures. However, research indicates that the gains in conceptual understanding do not come at the expense of understanding skills and procedures. Students who are taught using traditional curricula and students who are taught using reform curricula perform at about the same level when tested on skills and procedures.
Context vs. No Context	In general, throughout these research studies, students tended to do better in situations that covered topics and approaches that they had the opportunity to learn. For example, students who had been taught with a reform curriculum outperformed students who had been taught with a traditional curriculum on skills and procedures when they were embedded within contexts. Whereas, students who were taught using a traditional curriculum outperformed students who were taught using a reform curriculum on skills and procedures that appeared without a context.
Habits of the Mind	Students taught using problem-solving approaches are more likely to exhibit some of the Habits of Mind described in Table 4.3, such as positive disposition and perseverance, than those taught with traditional mathematics curricula.

Table 5.3 – Research Findings related to Problem-Solving Classrooms (Stein, Boaler, & Silver, 2003)

Achievement Linked to Instructional Practices	In a multi-year national study of middle school mathematics reform throughout economically disadvantaged communities, achievement was found to be linked to instructional practices. In those schools where the instructional practices were believed to be more problem-solving oriented, students' improvement in mathematics was greater than what it was for students who were in schools that were believed to consist of more routine skill-based practices (as measured by an assessment consisting of open-ended tasks). Additionally, students from these problem-solving oriented schools showed no decline in their abilities to perform routine calculations over the course of the study.
Task Complexity	In addition to the findings linked to instructional practices, researchers examined more subtle differences in teachers' instructional practices. When observing over 300 lessons from teachers who were trying to implement the recommendations in the NCTM standards, they found that the more complex the initial task was (e.g., requiring students to make and test conjectures, look for patterns), the more apt the task was to decline into a low-level activity (i.e., transformed from a higher depth of knowledge to a lower one – see Appendix A for information on Depth of Knowledge). In fact, students engaged in the intended cognitive processes in only 38% of tasks that were classified as "doing mathematics" tasks. This highlights the importance of the role of the teacher in the problem-solving process, as described in the previous section.

Flexible Thinking	In a different study of 300 high school students, ages 13 to 16 and tracked over 3-years, researchers found that students who were taught with reform curricula and learned through open-ended projects developed more flexible thinking and problem-solving skills than those students who were taught with traditional curricula. The students taught with the reform curricula outperformed the other group of students on a range of different assessments and the differences in scores between various subgroups of students (social class and gender) were not as pronounced.
Types of Questions and Depth of Knowledge	In a third study, this one at elementary school, researches found that student performance was linked to the complexity of the tasks (see Appendix A and the discussion on Depth of Knowledge) – with students who spent additional time on tasks explaining their solution methods and strategies learning more than those students who didn't have this opportunity.

Exercise 5.1 Take time, with colleagues, to review these research findings from their original sources and discuss the impact that they can have on your classroom instruction.

While the previous section introduced the problem-solving process, the next section will discuss some strategies for teaching problem solving.

Section 6: Teaching Problem-Solving Strategies

Throughout this unit of study, we have discussed methods for teaching problem-solving strategies, including the teachers role in setting up a problem-solving environment and fostering the development of important habits of mind. This section will allow you to explore some of those ideas further. The exercises in this section will take some time to complete and it is worth noting that the discussion with colleagues is an extremely important aspect of this section.

We begin by discussing some ways to model the problem-solving process in your classroom, as illustrated in Table 6.1. Subsequent exercises in this section will allow you to explore the idea of choosing rich problem-solving tasks for your classroom, along with ways to build an environment where students develop a positive disposition towards problem solving.

Table 6.1 – Modeling Problem-Solving
(Adapted from Schoenfeld, 1983)

Making the Process Transparent to Students	Making the process transparent to students requires going through all the various details of the problem, even when you, as the teacher, may find the steps immediate or obvious. I can often remember in graduate school sitting through lectures where my professors would proceed from one step to another stating that the move was obvious where it wasn't at all obvious to me. I would go home and write out the details between these steps and those details would often be multiple pages. I try to keep this in mind when working with my students. See Example 6.1 for an illustration of making the process transparent for students regarding Essential Question 11.
Using Students' Ideas	To create a successful problem-solving classroom environment, it is important to value students' ideas and allow them to build and construct their own knowledge. It is important for the teacher to moderate the conversations and help to guide students carefully to the solution. Asking "Why?" and "How do you know that?" are two important questions in the process, along with asking guiding questions like those

	described in Table 4.1. Recall, it will be important to decide which ideas are worth exploring and which maintain the construct of the initial tasks. You won't have time to explore all the good ideas generated. You may wish to assign some as projects.
Allowing Students to Challenge the Teacher	Schoenfeld describes a process where students are allowed to challenge the teacher. That is, if students are expected to employ problem-solving heuristics to tasks that they haven't seen, the teacher should be able to demonstrate the use of these heuristics to problems posed by the students as well. Schoenfeld begins his classes by asking whether or not students have any questions for him. In the beginning of the course students are somewhat reluctant to pose questions, but as the semester goes on, students open up to this idea. Employing this strategy can put you on the spot and you need to be prepared for it. However, it allows students to see you model the process. Just remember, if a student poses a question that you have seen before (which they often will), you are not truly engaging in the problem-solving process if you already know the solution.

Example 6.1 – Making the Process Transparent to Students

Take a moment to revisit Essential Question 11 from Section 3. When reading mathematics and presenting solutions, often the mathematics is presented in a concise format. The following summary is taken from Schoenfeld's discussion on this problem in *Problem Solving in the Mathematics Curriculum*, and highlights how to make the problem-solving process transparent for students (please see the source for further details).

> **Essential Question 11** Let $p(x)$ and $q(x)$ be polynomial functions with "reversed" coefficients. That is,
>
> $$\begin{aligned} p(x) &= a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x^1 + a_0 \\ q(x) &= a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-2} x^2 + a_{n-1} x^1 + a_n \end{aligned}, \text{ where } a_n \neq 0 \neq a_0$$
>
> What is the relationship between the zeros of $p(x)$ and $q(x)$? Explain.

(Adapted from Schoenfeld, 1983)

We might begin by asking ourselves how we should proceed on such a problem where we have no general method of finding the roots of these polynomials. Employing the look at easier related problem heuristic may lead us to look at the case where *p* and *q* are quadratics. We know that we can at least solve these. Considering

$$p(x) = ax^2 + bx + c$$

$$q(x) = cx^2 + bx + a$$

and using the quadratic formula, we find the roots of $p(x)$ and $q(x)$ to be

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b \pm \sqrt{b^2 - 4ac}}{2c}, \text{ respectively.}$$

While it is interesting to note that the numerators are the same, we don't necessarily see something that we can immediately generalize. So, we might either try the linear case to see what happens or proceed to try some specific examples for the quadratic case and see if we notice any patterns. Let's begin by looking at the linear case. In this case, we have

$$p(x) = ax + b$$

$$q(x) = bx + a.$$

And, the roots are $-\frac{b}{a}$ and $-\frac{a}{b}$, respectively. So, we notice that the roots are reciprocals of each other. We might wonder if this is true for the quadratic case as well. Rather than go back to our general solutions, it may be helpful to look at some specific examples. Why don't we choose some examples that are easy to factor. For example, if we start by choosing $p(x)$ as a quadratic in factored form it is easy to see its roots. For,

$$p(x) = (2x - 7)(3x + 4) = 6x^2 - 13x - 28,$$

its roots are 7/2 and –4/3. Furthermore, this choice for *p* defines *q* as

$$q(x) = -28x^2 - 13x + 6 = (-7x + 2)(4x + 3),$$

which has roots 2/7 and –3/4 which are the reciprocals of the roots of $p(x)$. Choosing another specific example will yield the same results and begins to convince us that this isn't just a coincidence. In fact, we notice that the factors are reversed. From here, we can develop the hypothesis that the roots of $p(x)$ are the reciprocals of the roots of $q(x)$. If we are not convinced, we might decide to try another quadratic example or some examples of cubic that can be easily factored.

Even though we have a nice hypothesis and can restate our problem (Let $p(x)$ and $q(x)$ be polynomials with reversed coefficients. Prove that the roots of $p(x)$ and $q(x)$ are reciprocals.), it may still be difficult to prove – not all polynomials are factorable over the integers, and keeping track of the coefficients could be tricky. At this point it is worthwhile to make certain that we understand our new problem statement. That is, what does it mean for some number r to be a root of $p(x)$? This would mean that $p(r) = 0$. Thus, given this information, it remains to be shown that $q(1/r) = 0$. In the general case, this may also prove to be difficult, so it may be worthwhile to revisit the quadratic case.

Let $p(x)$ and $q(x)$ be as before for the quadratic case. Then, if $p(r) = 0$, we have $ar^2 + br + c = 0$. And, $q\left(\frac{1}{r}\right) = c(\frac{1}{r})^2 + b\left(\frac{1}{r}\right) + a = \frac{c+br+ar^2}{r^2} = \frac{p(r)}{r^2} = 0$. From here, we should be able to see how the argument will generalize and be able to write up a nice proof. The proof will be rather brief, but it is the transparency of the above process that will allow students to receive insights into the problem. We see that we began by understanding the problem, examined special cases, looked for a pattern, restated the problem, and examined all the conditions of the problem.

Exercise 6.1 Think of some examples from your classroom where you make the problem-solving process transparent to your students. Share these examples and your strategies with your colleagues and record your thoughts in your journal.

Earlier we discussed the importance of choosing rich problems that allow students to explore content in-depth. The following adapted task selection criteria are based on those described by Grouws in *The Teacher's Role in Teaching Mathematics Through Problem Solving* and are based on the premise that tasks presented to students affect the mathematical skills that they acquire, affect the habits of mind they learn, and should require a high cognitive demand (see Appendix A for information on Depth of Knowledge and Cognitive Demand) to allow for a deep understanding of the mathematics studied (Grouws, 2003).

- Is the task centered on an important mathematical idea that is part of the curriculum?
- Is the task clearly stated and not unintentionally vague? [Remember, it is okay to have task that are vague and these can lead to many interesting discussion, but they should be intentionally vague to allow for the exploration of the content.]
- Does the task involve a context that students might find interesting?
- Can the task be solved with a wide variety of methods?
- Does the task lend itself to further exploration and have interesting extensions?

Exercise 6.2 will allow you to explore this concept further.

Exercise 6.2 Read *Introducing New Content Via Problems* in part C of the Annenberg Problem-Solving course at the following link:
http://www.learner.org/channel/courses/teachingmath/grades9_12/session_03/section_03_b.html .
Work with colleagues to create a problem that requires a solution that leads to the introduction of new content that could replace an area where you have previously defined a new concept and followed it with a series of examples.

Throughout this unit of study, we have discussed the importance of setting up a classroom environment where students are not afraid to take risks and where they develop a positive disposition towards mathematics and the problem-solving process. Exercise 6.3 will allow you to explore ways in which you can do this, including discussing choosing problems known as “low-threshold, high ceiling” tasks.

Exercise 6.3 Read through part C of the Annenberg Defining Problem-Solving course at http://www.learner.org/channel/courses/teachingmath/grades9_12/session_03/section_03_c.html and create some problems for your classes that could be described as “low-threshold, high ceiling” tasks. Share your tasks with your colleagues.

Section 7 will give you the opportunity to explore released items from the NECAP assessment and see how those items connect to the ideas in this unit of study.

Section 7: Examining NECAP Released Items and Student Work

This section contains exercises focused on items released from the New England Common Assessment Program, along with the sample student work (for constructed response items). Each released item is mapped to a primary Grade-Span Expectation as indicated by the codes in the released item support materials document. These Grade-Span Expectations can be located in Section 2. These exercises will allow you to explore how students approach some problems that connect to problem-solving, and provide the opportunity for you to make connections between these items and this unit of study. [It is important to note that much of problem-solving is done in a more meaningful way with local assessment than with large-scale assessment. So, while the exercises in this section allow you to explore some of the released NECAP items, you should be spending time examining your local assessments as well.]

Exercise 7.1 Locate all of the released NECAP items and practice test items, along with the released item support materials that deal with the GSEs highlighted in Section 2. Note, since these GSEs are local expectations, the NECAP items will not be coded to these particular GSEs. Thus, you will need to spend some time looking through the content and the task as well to determine the alignment. These items can be located at the following link: http://www.ride.ri.gov/Assessment/necap_math.aspx.

Exercise 7.2 For each of the multiple choice questions located in Exercise 7.1, determine multiple ways that students might obtain each of the answer options and how these methods connect to this unit of study.

Exercise 7.3 Read through the student work that is released for each of the constructed response items found in Exercise 7.1, and determine, based on the rubric provided, how the various responses were scored. Then, determine how the students' responses connect to this unit of study, including the various strategies used.

Exercise 7.4 Select some of the student responses available that illustrate some of the various problem-solving strategies highlighted in this unit of study and spend some time discussing how you might guide students to understand the problem, develop a plan, carry out the plan, and reflect on their solutions.

Exercise 7.5 Select some of the problems identified in Exercise 7.1 and reflect upon the tasks and determine other interesting questions that you might explore related to the tasks. For example, consider Practice Test Item #10 at the link given in Exercise 7.1. The correct response to this task is B. A further question to explore would be whether or not students can explain the change in concavity in the graph and be able to identify in the physical model where the graph should be concave up and where the graph should be concave down. Take a moment to reflect on this.

Exercise 7.6 Repeat the above exercises, but for your local assessments.

Section 8: Summary

This section contains four exercises focused on examining your current curricular materials and creating a classroom environment that is problem-solving oriented. This section is best completed with the collaboration of colleagues and requires extensive planning. The goal is to integrate the ideas presented in this unit of study into your lesson plans, instruction, and curricula.

Exercise 8.1 Examine your instructional materials and district curriculum and determine if they provide opportunities for students to test personal conjectures and emphasize a problem-solving approach when engaged in mathematical tasks.

Exercise 8.2 Examine your instructional materials and district curriculum and determine if they emphasize the development of important habits of the mind and contain meaningful tasks that students will find interesting.

Exercise 8.3 Examine your instructional materials and district curriculum and determine if they contain tasks that could be described as "low-threshold, high ceiling" tasks that will help foster the development of a positive disposition towards mathematics and flexible thinking.

Exercise 8.4 Create a curricular sequence/unit of study for a content topic in one of your courses that focuses on the problem-solving process and deepening students' understandings of the content through problem-solving. The sequence should consider all three principles of how students learn. The following table can serve as a model for your work.

Concept	Description of How Concept is Introduced	Target Depth of Knowledge Levels*	Description of Activities

*You may want to review Appendix A for information on Depth of Knowledge.

This unit of study was meant to introduce you to the problem-solving process and important habits of the mind that can have a direct impact on classroom instruction. It was intended to be used as a supplement to current curricula materials and provide an introduction to some of the research behind the problem-solving process.

If you are looking to build upon the ideas presented in this unit of study and looking to increase your depth of understanding of this vast topic, you are encouraged to explore the references listed in Section 10. In addition to the resources in Section 10 and those contained in exercises within the sections of this unit of study, you might want to explore the following sites (as of June 2008) for problem-solving activities and resources.

National Council of Teachers of Mathematics Illuminations Web Site
http://illuminations.nctm.org/

PBS Teachers
http://www.pbs.org/teachers/

Annenberg Media Learner.org
http://www.learner.org/

Balanced Assessment
http://balancedassessment.concord.org/

Federal resources for educational excellence
http://free.ed.gov/

National Library of Virtual Manipulatives
http://nlvm.usu.edu/en/nav/vlibrary.html

Journal of Recreational Mathematics
http://www.baywood.com/journals/PreviewJournals.asp?Id=0022-412x

Math Forum Problem of the Week Forum
http://mathforum.org/pow/

Park City Mathematical Institute Problem-Solving References & Resources
http://www.math.hmc.edu/~ajb/PCMI/problem_solve.html

Cut-the-Knot
http://www.cut-the-knot.org/index.shtml

Section 9: Answers to Exercises

Section 1: Purpose and Design

Exercise 1.1: Answers will vary.

Section 2: Connecting to the Grade-Level Expectations

Exercise 2.1: Primary GSEs are given on pp. 5–6 of Section 2.

Section 3: Essential Questions (Note: while sample solutions are given here, the problem-solving process details are left to the reader and should be discussed in groups.)

Essential Question 1: See Example 4.5.

Essential Question 2: See Example 4.2.

Essential Question 3: Sample solution: For any natural number n, consider the string of n natural numbers given by:

$$(n + 1)! + 2, (n + 1)! + 3, (n + 1)! + 4, \ldots, (n + 1)! + n, (n + 1)! + n + 1.$$

Then, $(n + 1)! + 2$ is divisible by 2 and hence not prime; $(n + 1)! + 3$ is divisible by 3 and hence not prime; $(n + 1)! + 4$ is divisible by 4 and hence not prime; …; $(n + 1)! + n + 1$ is divisible by $n + 1$ and hence not prime. Therefore, we have found a succession of n natural numbers that doesn't contain a prime.

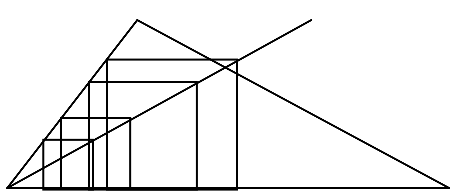
Essential Question 4: Since $2160 = 2^4 \cdot 3^3 \cdot 5$, there are $5 \times 4 \times 2 = 40$ positive divisors (to determine the total number of factors, note that you select anywhere from 0 to 4 factors of 2, 0 to 3 factors of 3, and 0 or 1 factor of 5).

Essential Question 5: $2\sqrt{5}$ inches (hint: make a net of the cube and keep the spider and fly at the same relative positions)

Essential Question 6: 1:5 (See http://jwilson.coe.uga.edu/squareF/square.html for a nice discussion of this problem and further explorations.)

Essential Question 7: 6, 8, 10 triangle & 5, 12, 13 triangle

Essential Question 8: See the process below which outlines a method for finding the square. Begin by relaxing one of the conditions and sketch squares where three vertices are on the triangle as shown and note that the locus of the fourth corner is on the straight line shown.



Essential Question 9: See Example 4.4.

Essential Question 10: part a: See Example 4.3; part b: $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = \left(\frac{1}{1!} - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{3!} - \frac{1}{4!}\right) + \ldots + \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right) = 1 - \frac{1}{(n+1)!}$

Essential Question 11: See Example 6.1.

Essential Question 12: red chip; explanations will vary, but writing out all the possibilities will reveal the answer.

Essential Question 13: about 7.5% (note: A common misconception is to ignore the base-rate information, i.e., that only 1% of women in this particular age group who get routine screening actually have cancer – this neglect usually leads to an estimation around 70% to 80%. It is worth noting that the probability of actually having cancer is proportional to this base rate.)

Essential Question 14: the answer depends upon the date on the pennies, but it may be close to 40% and you might determine whether or not your prediction seems accurate by performing a simulation and hypothesis test (see Example 6.6 in the *Probability & Counting Techniques* unit of study). You might want to think about why the expectation isn't 50% heads.

Essential Question 15: Answers will vary. Sample answer: Minimizing the sum of the absolute values of the residuals does not lead to a unique solution. For example, consider finding the line of best fit for the points (0, 0), (0, 2), (2, 2), and (2, 4). Any line that passes between the two points at $x = 0$ and between the two points at $x = 2$ will result in a sum of the absolute values of the residuals being 4 (which is the minimum value that can be obtained – which can be seen by trying other lines that either are outside of both points at $x = 0$ and $x = 2$ or inside one set and outside of the other). Note, that the sum of the squares of the residuals will be different for each of the lines described above and will be minimum for the line $y = 1 + x$. It may also be helpful to graph functions which are the sum of multiple absolute value functions and functions that are the sum of multiple squared functions and note where minimums occur.

Essential Question 16: Yes, sample explanation: Begin by rowing Bill over to the other side of the river, leaving Bob and the Burgers on one side. Row back and pick up the bag of burgers. Bring them to the other side, but bring Bill back in your boat. One to the original side, row Bob over to the other side, leaving him alone with the burgers again. Now go back and get Bill.

Essential Question 17: Solution for 9 coins: begin by weighing two sets of 3 coins. This one weighing will determine one of three coins that is the counterfeit coin (either in one of the weighed sets or determined to be on the "set-aside" set if the two weighed sets balance). Now, weigh 2 of these 3 coins. This will allow you to determine the counterfeit coin in two weighings.

Sections 4 through 8 – Answers will vary for the exercises in these sections.

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Appendix A – Implied Cognitive Demand and Depth of Knowledge

A fundamental criterion used to develop the NECAP GLEs and GSEs is that the expectations should explicitly indicate cognitive demand (how content interacts with process) and that there should be a mix of cognitive demand levels at all grades. That is, one should not assume that students at lower grades do less cognitively demanding work. The cognitive demand or depth of knowledge required by an expectation or an assessment item is related to the number and strength of connections of concepts and procedures that a student needs to make to produce a response, including the level of reasoning required along with self-monitoring. Furthermore, there are additional factors that influence cognitive demand including contextual requirements, language, the number and variety of representations, requirements for generalizations to new situations, and the opportunity to learn.

It is important to note that depth of knowledge is not synonymous with difficulty. As an example, solving a multi-step linear equation with variables on both sides may be a difficult task for middle school students; however, the task can be solved by applying a standard procedure making the task of low complexity.

The NECAP states believe that expectations and assessment should be aligned in terms of their cognitive complexity. That is, the cognitive complexities of the assessment items should match that of the standards (what students are expected to know and be able to do). To ensure this alignment, the NECAP states have adopted Norman L. Webb's (senior researcher with the Wisconsin Center for Educational Research) Depth of Knowledge classification system. Norman Webb's system is based on four levels of classification. The full descriptions of each level are given on pages 49-50. The levels can be summarized as follows.

Level 1	Recall
Level 2	Skill/Concept
Level 3	Strategic Thinking
Level 4	Extended Thinking

The NECAP states, together with a committee of educators, analyzed the GLEs and GSEs for their implied cognitive demand. That is, all aspects of each expectation were analyzed and the implied cognitive demand levels were recorded. One of the charges of the NECAP test item review committees is to ensure that assessment items align not only with the expectations but also with their implied cognitive demands. The range of cognitive demands for each GLE and GSE is summarized in Table 1 on page 2. It should be noted that the highest level listed for each GLE and GSE should be thought of as a "ceiling" not a "target". That is, the goal is to write items which cover the range of the levels indicated and not just the highest level. If one assesses only at the "target" level, all GLEs with a level 3 (for example) as their highest cognitive demand would only be assessed at level 3. This would potentially have two negative impacts on the assessment: 1) The assessment as a whole would be too difficult; and 2) important information about student learning along the achievement continuum would be lost. To the extent possible, GLEs and GSEs should be assessed at the “ceiling” and at least one level below the “ceiling” in order to provide additional diagnostic information to educators. Furthermore, Table 2 shows an example of an expectation and how the different aspects of the expectation interact with Table 1.

Table 1

	Depth of Knowledge Levels for NECAP Assessment						
	2	**3**	**4**	**5**	**6**	**7**	**10**
M(N&O)–X–1	1, 2	1, 2	1, 2	1, 2	1, 2	1, 2	
M(N&O)–X–2	1	2	2	2	2	2	1, 2, 3
M(N&O)–X–3	1, 2	2	2	2,3	2,3		
M(N&O)–X–4		1, 2, 3	1, 2, 3	1, 2, 3	1, 2, 3	1, 2, 3	1, 2, 3
M(N&O)–X–5	1, 2						
M(G&M)–X–1	1, 2, 3	1, 2	1, 2	1, 2	1, 2		
M(G&M)X–2						1, 2	1, 2, 3
M(G&M)–X–3			1, 2	1, 2	1, 2		
M(G&M)–X–4				1, 2		1, 2	2, 3
M(G&M)–X–5			1, 2		1, 2	1, 2, 3	1, 2, 3
M(G&M)–X–6	1, 2	1, 2	1, 2	1, 2	1, 2, 3	1, 2, 3	1, 2, 3
M(G&M)–X–7	This GLE will NOT be directly assessed but embedded in problems in other content strands.						1, 2
M(G&M)–X–8							
M(G&M)–X–9							2, 3
M(F&A)–X–1	2	2	2	2	2, 3	2, 3	2, 3
M(F&A)–X–2					1, 2	1, 2, 3	1, 2, 3
M(F&A)–X–3			1	1	1, 2	1, 2	1, 2
M(F&A)–X–4	1	1, 2	1, 2	1, 2	1, 2	1, 2	
M(DSP)–X–1	1, 2, 3	1, 2, 3	1, 2, 3	1, 2, 3	1, 2, 3	1, 2, 3	2, 3
M(DSP)–X–2	2, 3	2, 3	2, 3	2, 3	2, 3	2, 3	2, 3
M(DSP)–X–3		1, 2		1, 2		2, 3	1, 2, 3
M(DSP)–X–4	2		2, 3		2, 3		1, 2, 3
M(DSP)–X–5		1, 2	1, 2	1, 2	1, 2, 3	1, 2, 3	1, 2, 3

Black cells indicate GLEs or GSEs that are not assessed on NECAP at the given level.

Sample Mathematics GLE* for End of Grade 6	Potential DoK Levels	DoK Ceiling	Aspects of GLE at different levels**
M(F&A)–6–1 **Identifies and extends to specific cases a variety of patterns** (linear and nonlinear) represented in models, tables, sequences, graphs, or in problem situations; or writes a rule in words or symbols for finding specific cases of a linear relationship; or writes a rule in words or[sc] symbols for finding specific cases of a nonlinear relationship; and writes an expression or[sc] equation using words or[sc] symbols to express the **generalization** of a linear relationship (e.g., twice the term number plus 1 or[sc] $2n + 1$).	**2, 3**	**3**	Level 2 **Extends a pattern to a specific case** **Level 3** Generalizes a pattern

Table 2

*GLE NOTES: Underlining in the GLE indicates that this concept or skill is “new” to grade 6 for assessment purposes. The superscript “sc” indicates that students have a choice in how they complete the task (e.g., students can use words **or** symbols to express the rule).

**Recall, one must also consider other factors when making decisions on Depth of Knowledge levels such as contextual requirements, language, the number and variety of representations, requirements for generalizations to new situations, and the opportunity to learn.

Depth of Knowledge Descriptors for Mathematics
Norman L. Webb
March 28, 2002

Mathematics Depth of Knowledge Levels

Level 1 (Recall) includes the recall of information such as a fact, definition, term, or a simple procedure, as well as performing a simple algorithm or applying a formula. That is, in mathematics a one-step, well-defined, and straight algorithmic procedure should be included at this lowest level. Other key words that signify a Level 1 include “identify,” “recall,” “recognize,” “use,” and “measure.” Verbs such as “describe” and “explain” could be classified at different levels depending on what is to be described and explained.

Level 2 (Skill/Concept) includes the engagement of some mental processing beyond a habitual response. A Level 2 assessment item requires students to make some decisions as to how to approach the problem or activity, whereas Level 1 requires students to demonstrate a rote response, perform a well-known algorithm, follow a set procedure (like a recipe), or perform a clearly defined series of steps. Keywords that generally distinguish a Level 2 item include “classify,” “organize,” ”estimate,” “make observations,” “collect and display data,” and “compare data.” These actions imply more than one step. For example, to compare data requires first identifying characteristics of the objects or phenomenon and then grouping or ordering the objects. Some action verbs, such as “explain,” “describe,” or “interpret” could be classified at different levels depending on the object of the action. For example, if an item required students to explain how light affects mass by indicating there is a relationship between light and heat, this is considered a Level 2. Interpreting information from a simple graph, requiring reading information from the graph, also is a Level 2. Interpreting information from a complex graph that requires some decisions on what features of the graph need to be considered and how information from the graph can be aggregated is a Level 3. Caution is warranted in interpreting Level 2 as only skills because some reviewers will interpret skills very narrowly, as primarily numerical skills, and such interpretation excludes from this level other skills such as visualization skills and probability skills, which may be more complex simply because they are less common. Other Level 2 activities include explaining the purpose and use of experimental procedures; carrying out experimental procedures; making observations and collecting data; classifying, organizing, and comparing data; and organizing and displaying data in tables, graphs, and charts.

Mathematics Depth of Knowledge Levels continued

Level 3 (Strategic Thinking) requires reasoning, planning, using evidence, and a higher level of thinking than the previous two levels. In most instances, requiring students to explain their thinking is a Level 3. Activities that require students to make conjectures are also at this level. The cognitive demands at Level 3 are complex and abstract. The complexity does not result from the fact that there are multiple answers, a possibility for both Levels 1 and 2, but because the task requires more demanding reasoning. An activity, however, that has more than one possible answer and requires students to justify the response they give would most likely be a Level 3. Other Level 3 activities include drawing conclusions from observations; citing evidence and developing a logical argument for concepts; explaining phenomena in terms of concepts; and using concepts to solve problems.

Level 4 (Extended Thinking) requires complex reasoning, planning, developing, and thinking most likely over an extended period of time. The extended time period is not a distinguishing factor if the required work is only repetitive and does not require applying significant conceptual understanding and higher-order thinking. For example, if a student has to take the water temperature from a river each day for a month and then construct a graph, this would be classified as a Level 2. However, if the student is to conduct a river study that requires taking into consideration a number of variables, this would be a Level 4. At Level 4, the cognitive demands of the task should be high and the work should be very complex. Students should be required to make several connections—relate ideas *within* the content area or *among* content areas—and have to select one approach among many alternatives on how the situation should be solved, in order to be at this highest level. Level 4 activities include designing and conducting experiments; making connections between a finding and related concepts and phenomena; combining and synthesizing ideas into new concepts; and critiquing experimental designs.

References

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